RESEARCH ARTICLE

Model predictive control for tracking randomly varying references

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This paper proposes a model predictive control scheme for tracking a-priori unknown references varying in a wide range and analyzes its performance. It is usual to assume that the reference eventually converges to a constant in which case convergence to zero of the tracking error can be established. In this note we remove this simplifying assumption and characterize the set to which the tracking error converges and the associated region of convergence.

Keywords: Tracking control; Nonlinear Model predictive control; constrained nonlinear systems

1 Introduction

Recent advances in Model Predictive Control (MPC) have enabled MPC to deal with control problems of large scale systems and high speed applications. These include electromechanical, power systems and telecommunication problems. In these areas, it is no longer true that the reference signal can be considered constant, or even piecewise constant. When the reference is time varying it is customary to assume exact knowledge of the trajectory to be tracked. In practice the reference signal is commonly affected by uncertainties which can even be large. For instance, in automotive engineering, the desired torque of spark ignition engines is obtained by controlling airflow and spark timing (Cairano et al. 2012). Due to limited spark timing authority, the airflow-generated torque must be controlled in a range around the requested torque. In hybrid electric vehicles the combustion engine power needs to be controlled so that the difference with the driver-requested power can be achieved by electric power (Cairano et al. 2013). The controller needs to track the engine speed reference. To maximize wind power extraction, a variable-speed wind turbine should operate as close as possible to its optimal power coefficient. The generator torque is used as a control input to improve wind energy capture by forcing the wind turbine to stay close to the maximum energy point which depends on the wind speed (Pao and Johnson 2011). In general, ignoring the dynamical and stochastic aspect of wind, leads to significant power losses (Boukhezzar et al. 2006). In power systems it is of great interest to reduce demand supply imbalances in the grid. More specifically, it is relevant to design control systems which enable a plant to track time varying references representing the energy imbalance (Hindi et al. 2011). In this context the reference has a random component that can effectively be modeled as a zero mean random walk. Uncertainty in the reference signal arises, as well, in distributed discrete-time coordinated tracking problems where a team of vehicles communicating with their local neighbours at discrete-time instants tracks a time-varying reference (Cao et al. 2009). When a dynamic system, linear or nonlinear, is required to track a time varying reference it may be possible to achieve asymptotic or exponential stability of a robust invariant set. Establishing such stability is often difficult and it is common, therefore, to assume that the reference signal converges to a constant; this assumption simplifies the analysis considerably and usually permits asymptotic or exponential stability to be established. In the present paper we

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investigate performance, in the context of MPC, when this simplifying assumption is *not* made and the reference can vary arbitrarily over a wide range. MPC schemes facing the problem of tracking piece-wise constant references for linear systems have been discussed in Limon et al. (2010, 2008), Chisci and Zappa (2003), Pannocchia and Rawlings (2003), Rawlings and Mayne (2009), Maeder and Morari (2010), Blanchini and Miani (2000) and the references therein. The tracking problem for nonlinear systems was originally tackled by the reference governor approach Bemporad (1998), Angeli and Mosca (1999), Gilbert and Kolmanovsky (2002). In Magni et al. (2001) an internal model of the exosystem was employed to enable output MPC to track exogenous systems. In Chisci et al. (2005) the tracking problem has been approached by embedding the nonlinear system in a set of Linear Parameter Varying models and exploiting extended constraint admissible invariant sets while in Ferramosca et al. (2009) feasibility and convergence of the MPC controller for tracking piece-wise constant references is guaranteed by introducing an artificial reference as decision variable and minimizing the tracking error and the deviation between the desired and the artificial steady state. Recently in Fagiano and Teel (2012) the tracking problem has been discussed introducing a generalized terminal state constraint. MPC of constrained systems when the reference trajectory is known a-priori is discussed in Faulwasser and Findeisen (2011), Limon et al. (2012), Falugi and Mayne (2013). The relationship between optimal control, nonlinear MPC and control Lyapunov function is discussed in Primbs et al. (2000).

In this paper the feasibility of the problem for any desired admissible steady state is guaranteed by adding an artificial target (Limon et al. 2010, Ferramosca et al. 2009) while confinement of the trajectories when the reference is varying in a compact set is guaranteed by an additional constraint bounding a control Lyapunov function. Preliminary results have been presented in Falugi and Mayne (2012) while in Falugi and Mayne (2013) a simplified scheme suitable for tracking a random reference belonging to a sufficiently small set has been proposed. The approach proposed in the paper guarantees convergence to the desired reference when it remains constant and provides a characterization of the set in which the tracking error lies when the reference varies.

The paper is organized as follows. In section 2 the problem of tracking randomly varying references is discussed and a model predictive control scheme achieving the confinement of the state trajectory in a region is proposed. In section 3 the feasibility and the convergence of the implemented receding horizon strategy is examined. In section 4 we illustrate the performance by means of a simulation example and finally conclusions are given in section 5.

1.1 Notation and basic definitions

Let $\mathbb{R}$ denote the field of real numbers and $\mathbb{R}^m$ the m -dimensional Euclidean space, $\mathbb{I}_{\geq 0}$ the field of nonnegative integers and for $j, k \in \mathbb{I}_{\geq 0}$, the integer set $\mathbb{I}_{j,k} \triangleq \{j, j+1, \dots, k\}$. In Euclidean space, the length of a vector x is denoted by $|x|$, the closed unit ball $\{x \mid |x| \leq 1\}$ by $\mathcal{B}$ and $|x|_Q \triangleq \sqrt{x'Qx}$ with $Q = Q' \in \mathbb{R}^{n \times n}$ and $Q > 0$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function if $\alpha(0) = 0$ and $\alpha(s)$ is strictly increasing for all $s > 0$. It is furthermore a class $\mathcal{K}_{\infty}$ if $\alpha(s) \rightarrow \infty$ as $s \rightarrow \infty$. Given the state z at time k , z^+ denotes the state at the next time instant.

2 The tracking problem

Consider the following prototype tracking problem (Rawlings and Mayne 2009, page 159) where the system has an unstabilizable component. The system is described by

$$\begin{aligned}x^+ &= f(x, u) \\ r^+ &= r + \xi \\ y &= h(x)\end{aligned}$$

where $f(\cdot)$ and $h(\cdot)$ are twice continuously differentiable, x is the state of the system being controlled, u is the control input, r is the output reference and ξ is an unknown signal that takes values in the compact set Ξ containing the origin and is such that r remains in a compact set $\mathcal{R}$; the output y is required to track the reference signal r . If r is constant, the control is required to steer the initial state of the system to an equilibrium state $x_s(r)$ satisfying

$$\begin{aligned}x_s(r) &= f(x_s(r), u_s(r)) \\ r &= h(x_s(r)).\end{aligned}\tag{1}$$

The output reference r , however, may vary arbitrarily within $\mathcal{R}$; r may be regarded as a discrete state Markov process and does not necessarily converge to a constant. A composite state may be defined as (x, r) ; the state component r is not stabilizable. The target state and control $(x_s(r), u_s(r))$, associated with the reference r , may be obtained by solving a simple optimization problem. It is assumed that the system is subject to state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ respectively, where $\mathbb{X}$ is a closed subset of $\mathbb{R}^n$ and $\mathbb{U}$ a compact subset of $\mathbb{R}^m$.

MPC is a widely used control technique and there exist many theoretical results guaranteeing regulation to an assigned steady state for constrained systems. However, when the reference changes, the feasibility of a standard MPC formulation can be lost and stability of the closed loop system can not be guaranteed. Here feasibility of the MPC algorithm for changing references is guaranteed using an artificial reference as decision variable, as in Ferramosca et al. (2009), while the confinement of the state trajectory is achieved by the introduction of a constraint which is activated only when the controlled system is evolving in a neighbourhood of the admissible equilibrium states.

Given the current state x and actual reference r , the model predictive control action is obtained by solving a finite horizon optimal control problem introducing an artificial reference r_a as decision variable (in addition to $\mathbf{u}$). Because future values of the reference are unknown, it is customary to use estimates in place of actual values; in this case, we employ a constant prediction of the future value of r and the artificial reference r_a to compute the controller; the function $V_N(x, r_a, \mathbf{u})$ costs the deviation of the state and control trajectories from the constant set point $(x_s(r_a), u_s(r_a))$ with $r_a \in \mathcal{R}$:

$$V_N(x, r_a, \mathbf{u}) = V_f(\tilde{x}(N; x, r_a, \mathbf{u}), r_a) + \sum_{i=0}^{N-1} \ell(\tilde{x}(i; x, r_a, \mathbf{u}), \tilde{u}(i; x, r_a, \mathbf{u}))$$

where, for each $i \in \mathbb{I}_{\geq 0}$, $\tilde{x}(i; x, r_a, \mathbf{u}) \triangleq x(i; x, \mathbf{u}) - x_s(r_a)$, $\tilde{u}(i; x, r_a, \mathbf{u}) \triangleq u(i) - u_s(r_a)$ and $x(i; x, \mathbf{u})$ is the solution of $x^+ = f(x, u)$ at time i if the initial state $x(0) = x$ and the control sequence is $\mathbf{u} \triangleq \{u(0), u(1), \dots, u(i)\}$. The on-line optimization problem $\mathbb{P}_N(\cdot)$ is the minimization of $J_N(\cdot)$ subject to $(u, r_a) \in \mathcal{U}_N(\cdot)$ where $J_N(\cdot)$ and $\mathcal{U}_N(\cdot)$ are defined by

$$J_N(x, r_c, c, r_a, \mathbf{u}) \triangleq V_N(x, r_a, \mathbf{u}) + |r_c - r_a|_L^2$$

and

$$\mathcal{U}_N(x, c) \triangleq \{(\mathbf{u}, r_a) \mid x(i) \in \mathbb{X}, u(i) \in \mathbb{U} \forall i \in \mathbb{I}_{0, N-1}, \\ x(N) \in \mathbb{X}_f(r_a), V_N(x, r_a, \mathbf{u}) \leq c, r_a \in \mathcal{R}\}$$

where $L = L' > 0$, the function $V_f(\cdot)$ is a control Lyapunov function and $\mathbb{X}_f(r) \triangleq \{x \mid V_f(x - x_s(r), r) \leq \alpha\}$ for some $\alpha > 0$ is the associated control invariant set. Note the appearance of two new parameters (or pseudo states) c and r_c . The parameter c determines the size of the region where we want to confine the trajectories of the controlled system and in general it can depend on r . Since the reference r varies randomly and its future values are unknown it is important to perform the tracking only when the controlled system lies in the region where we want confine the trajectories. If we change the reference when the system is still far from the equilibrium states $x_s(r)$ with $r \in \mathcal{R}$ we can experience the appearance of undesired overshoots as illustrated in the simulation results. Minimization of $J_N(x, r_c, c, r_a, \mathbf{u})$ with respect to $(\mathbf{u}, r_a)$ yields $J_N^0(x, r_c, c)$, $\mathbf{u}^0(x, r_c, c)$, $r_a^0(x, r_c, c)$ and

$$V_N^0(x, r_c, c) \triangleq V_N(x, r_a^0(x, r_c, c), \mathbf{u}^0(x, r_c, c)) \quad (2)$$

where the implicit model predictive control law is $\kappa_N(\cdot)$ defined by

$$\kappa_N(x, r_c, c) = u^0(0; x, r_c, c) \quad (3)$$

where $u^0(0; x, r_c, c)$ is the first element of the sequence $\mathbf{u}^0(x, r_c, c)$. The updating equations are:

$$x^+ = f(x, u^0(0; x, r_c, c)) \quad (4)$$

$$r^+ = r + \xi \quad (5)$$

$$r_c^+ = \begin{cases} r_c & \text{if } V_N^0(x, r_c, c) \geq c^*(r + \xi) \\ r + \xi & \text{otherwise} \end{cases} \quad (6)$$

$$c^+ = \begin{cases} \gamma & \text{if } V_N^0(x, r_c, c) \geq c^*(r + \xi) \\ c^*(r + \xi) & \text{otherwise} \end{cases} \quad (7)$$

with initial conditions

$$x(0) = x_0$$

$$r(0) = r_0 \in \mathcal{R}$$

$$r_c(0) = \text{any } r \text{ in interior of } \mathcal{R}$$

$$c(0) = \gamma$$

where $\gamma \geq \max_{r \in \mathcal{R}} c^*(r)$ must be chosen sufficiently large so that the constraint $V_N(x, r_a, \mathbf{u}) \leq c$ is inactive at the first time instant. The function $c^* : \mathbb{R} \rightarrow \mathbb{R}$, mapping the reference value r to $c^*(r)$ with domain $\mathcal{R}$, is precalculated as described later. A lower bound $\bar{\gamma}$ of γ can be estimated off-line by selecting a large sample of significant $x_0 \in \mathbb{X}$ and solving the optimization problem $\mathbb{P}_N(x, r_c, c)$ while omitting the constraint $V_N(x, r_a, \mathbf{u}) \leq c$. The initial condition $r_c(0)$ can be set to be any value in the interior of $\mathcal{R}$. The choice is dictated by the following reasons: if x_0 is distant from the set of equilibrium states it is meaningful to choose a reference $r_c(0)$ associated with a steady state that is in the center of the set of steady states since the future evolution of r is unknown. Contrarily, if x_0 is close to the set of steady states the choice $r_c(0) = r(0)$ is more appropriate. While $|(x(t), u(t)) - (x_s(r), u_s(r))|$ is large, $r_c(t)$ defined in (6) is kept constant and when $(x(t), u(t))$ lies in a neighbourhood of $(x_s(r), u_s(r))$, specified by the constraint

$V_N(x, r_a, \mathbf{u}) \leq c$, $r_c(t) = r(t)$ where r is arbitrarily varying in $\mathcal{R}$. This choice is motivated by the simplicity and robustness of the algorithm. Indeed, if $r_c(t) = r(t)$, when $(x(t), u(t))$ is far from $(x_s(r), u_s(r))$, a time varying reference can generate undesirable overshoots as illustrated in the example.

The constraint $V_N(x, r_a, \mathbf{u}) \leq c$ ensures that the artificial reference can change at time t only if the cost $V_N(\cdot)$ does not exceed the bound $c = c(t)$ thereby guaranteeing that the trajectory will not leave a possibly small region of the state space denoted by $X^{\mathcal{R}}$. The region $X^{\mathcal{R}}$ defined below is such that $x_s(r) \in X^{\mathcal{R}}$ for all $r \in \mathcal{R}$ and guarantees viability from one equilibrium state to another. The value of c depends on the cost $V_N(x, r_a, \mathbf{u})$ at the previous step and can not be smaller than a certain value. To specify c further analysis is required.

The terminal ingredients $V_f(\cdot)$ and $\mathbb{X}_f(r)$ are chosen to achieve stability of the controlled system and they can be determined using the linearization of $f(\cdot)$ at $(x_s(r), u_s(r))$ (see Rawlings and Mayne (2009), Ferramosca et al. (2009), Zeilinger et al. (2010)). In particular, we assume:

Assumption 2.1 For each $r \in \mathcal{R}$ the linearized system is controllable at $(x_s(r), u_s(r))$.

The pair $(V_f(\cdot, r), \mathbb{X}_f(r))$ is defined using the usual stability assumption:

Assumption 2.2 There exist functions $k_f(x, r) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$, $V_f(x, r) : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}_{\geq 0}$ and two $\mathcal{K}_{\infty}$ functions α_1 and α_2 such that for all $r \in \mathcal{R}$ and all $x \in \mathbb{X}_f(r)$

- $k_f(x, r) \in \mathbb{U}$
- $f(x, k_f(x, r)) \in \mathbb{X}_f(r)$
- $\alpha_1(|\tilde{x}(r)|) \leq V_f(\tilde{x}(r), r) \leq \alpha_2(|\tilde{x}(r)|)$
- $V_f(f(x, k_f(x, r)) - x_s(r), r) \leq V_f(x - x_s(r), r) - \ell(x - x_s(r), k_f(x, r) - u_s(r))$.

Moreover we assume that for all $r \in \mathcal{R}$ there exists a neighbourhood $\mathcal{N}_r$ of $x_s(r)$ such that for all $x \in \mathcal{N}_r$, the function $V_f(x, r)$ is twice continuously differentiable.

Note that Assumption 2.2 implicitly requires that for each $r \in \mathcal{R}$ the set $\mathbb{X}_f(r)$ is control invariant and $V_f(\cdot)$ is the associated local Lyapunov function. Let $\tilde{x}(r) \triangleq x - x_s(r)$ and $\tilde{u}(r) \triangleq u - u_s(r)$. In order to use $V_N^0(x, r_c, c)$ and $J_N^0(x, r_c, c)$ as Lyapunov functions we further require that ℓ satisfies

Assumption 2.3 There exists a $\mathcal{K}_{\infty}$ function α_3 such that for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ it holds $\ell(\tilde{x}(r), \tilde{u}(r)) \geq \alpha_3(|\tilde{x}(r)|)$ for all $r \in \mathcal{R}$. Moreover we assume that for all $r \in \mathcal{R}$ the function $\ell(\cdot, \cdot)$ is twice continuously differentiable in a neighbourhood of $(x_s(r), u_s(r))$.

A constant reference output r is said to be admissible if $\mathbb{X}_f(r) \subset \mathbb{X}$ and if $u_s(r) + \varepsilon \mathcal{B} \in \mathbb{U}$ where $\varepsilon > 0$ is a fixed sufficiently small real.

Assumption 2.4 All $r \in \mathcal{R}$ are admissible.

Even if $f(\cdot)$ and $h(\cdot)$ are twice continuously differentiable it can happen that the functions $x_s(r)$ and $u_s(r)$ satisfying (1) are not Lipschitz continuous. For instance the family of equilibrium points for $x_1^+ = x_1 + x_1^3 - x_2$, $x_2^+ = 2x_2 - u$, $y = x_2$ satisfy $x_{2,s}(r) = r$, $u_s(r) = r$ and $x_{1,s}(r) = (r)^{1/3}$; $x_{1,s}(r)$ is not Lipschitz continuous at $r = 0$. Lipschitz continuity of a cost function with respect to r is used in Ferramosca et al. (2009) to establish that a constant reference can be tracked. In particular, using the quadratic functions $\ell(x, u) \triangleq |x|_Q^2 + |u|_R^2$ and $V_f(x, r_a) \triangleq |x|_{P_{r_a}}^2$ and exploiting quadratic bounds with respect to variation of r_a , it is possible to prove that r_a tends to r . For this reason we assume

Assumption 2.5 The functions $x_s(r) : r \rightarrow \mathbb{X}$ and $u_s(r) : r \rightarrow \mathbb{U}$ defined by (1) are Lipschitz continuous for all $r \in \mathcal{R}$.

If Assumption 2.5 is not satisfied, one way to overcome this difficulty is to employ (x_s, u_s) as a decision variable and to add $x_s = f(x_s, u_s)$, $r = h(x_s)$ as additional constraints to the optimization problem. In this case the computational burden increases.

We now introduce the main assumptions and ingredients that are required to solve the tracking problem when r takes arbitrary values in $\mathcal{R}$. The tracking problem requires that viability between reference states is preserved; we characterize the property by means of level sets. For each $(r, \vartheta) \in \mathcal{R} \times \mathbb{R}_{\geq 0}$, let $\mathcal{L}(r, \vartheta) \triangleq \{x \mid V_N^0(x, r, \gamma) \leq \vartheta\}$, a sublevel set of $V_N^0(\cdot, r, \gamma)$ defined in (2). For each $r \in \mathcal{R}$, let the variable $c^*(r)$ employed in (6)-(7) be defined by

$$c^*(r) \triangleq \min_{\vartheta \in \mathbb{R}_{\geq 0}} \{\vartheta \mid x_s(r') + \varepsilon \mathcal{B} \in \mathcal{L}(r, \vartheta) \ \forall \ r' \in \mathcal{R}\}. \quad (8)$$

where $\varepsilon > 0$ is any fixed sufficiently small real. Notice that $0 < c^*(r) < +\infty$ and $c^*(r)$ can be approximately evaluated solving the optimization problem $\mathbb{P}_N(x, r_c, \gamma)$ with a representative sample of initial conditions $x_s(r')$, $r' \in \mathcal{R}$ where $r_c = r$ and adding a margin. The evaluation of $c^*(r)$ obtained by solving $\mathbb{P}_N(x_s(r'), r, \gamma)$ has to be performed making sure that the constraint $V_N(x, r_a, \mathbf{u}) \leq \gamma$ is not active. Indeed, since $c^*(r)$ is precalculated, the problem $\mathbb{P}_N(\cdot)$ can be solved omitting the constraint $V_N(x, r_a, \mathbf{u}) \leq \gamma$.

Let $X_r \triangleq \mathcal{L}(r, c^*(r))$ denote the associated sublevel sets of $V_N^0(\cdot, r, \gamma)$ where γ is the value of $c(0)$ (see (7)); thus, for each r , $r' \in \mathcal{R}$, $x_s(r) \in X_{r'}$. Finally, let

$$\mathcal{X}_N(\gamma) \triangleq \{x \mid \mathcal{U}_N(x, \gamma) \neq \emptyset\} \quad (9)$$

$$X^{\mathcal{R}} \triangleq \cup_{r \in \mathcal{R}} X_r \quad (10)$$

and

Assumption 2.6 The inclusion $X^{\mathcal{R}} \subset \mathcal{X}_N(\gamma)$ holds.

The sets introduced above are illustrated in Figure 1. At any time $t > 0$, the choice of r_c and

Figure 1.: The sets $X_s^{\mathcal{R}} = \cup_{r \in \mathcal{R}} x_s(r)$, $X_r = \mathcal{L}(r, c^*(r))$, $X^{\mathcal{R}}$ and the point $x_s(r)$

c as specified in (6-7) ensures (as shown below) that $X^{\mathcal{R}}$ is invariant for the closed loop system and that the solution $x^+ = f(x, \kappa_N(x, r_c, c))$ enters $X^{\mathcal{R}}$ when $x_0 \in \mathcal{X}_N^{\mathcal{R}}(\gamma) \setminus X^{\mathcal{R}}$.

3 Analysis

When the measured state does not lie in $X^{\mathcal{R}}$ the value of c is chosen sufficiently large to ensure the constraint $V_N(x, r_a, \mathbf{u}) \leq c$ is never active; the dynamic parameter r_c defined in (6) ensures that the trajectories approach and enter $X^{\mathcal{R}}$ without abrupt changes due to arbitrary variations of the reference. Feasibility of $\mathbb{P}_N(x, r_c, c)$ is guaranteed for all $t > 0$ if $\mathbb{P}_N(\cdot)$ is feasible at $t = 0$:

Proposition 3.1: *Suppose Assumptions 2.2 and 2.3 are satisfied. If $\mathbb{P}_N(x, r_c, c)$ is feasible then $\mathbb{P}_N(x^+, r_c^+, c^+)$ is feasible where $x^+ = f(x, \kappa_N(x, r_c, c))$ and r_c^+ , c^+ satisfy equations (6-7).*

Proof: - See the appendix.

The proof of convergence requires the technical notion of a weakly invariant set

Definition 3.2: (Kloeden and Marín-Rubio 2003) A nonempty compact set Ω is weakly positively invariant for a set valued map $x^+ \in g(x)$ if for each $x_0 \in \Omega$ there exists a solution $\varphi(\cdot; x_0)$, satisfying $\varphi(t+1; x_0) \in g(\varphi(t; x_0)) \forall t \geq 0$, such that $\varphi(t; x_0) \in \Omega$ for all $t \geq 0$.

The proposed approach enjoys the following property

Proposition 3.3: Suppose Assumptions 2.1-2.6 are satisfied. If r is constant ($\xi \equiv 0$), the equilibrium pair $(x_s(r), u_s(r))$ is attractive for the controlled system $x^+ = f(x, \kappa_N(x, r_c, c))$ with a region of attraction $\mathcal{X}_N(\gamma)$.

Proof: - See the appendix.

Remark 1: It is possible to prove that the equilibrium pair $(x_s(r), u_s(r))$ is asymptotically stable using continuity of the value function $J_N^0(x, r_c, c)$ at $x_s(r)$ for all $r \in \mathcal{R}$ and following arguments similar to those in Meadows et al. (1995). Continuity of the value function $J_N^0(x, r_c, c)$ at $x_s(r)$ for all $r \in \mathcal{R}$ [see Theorems C.28 in Rawlings and Mayne (2009)] is guaranteed by continuity of $J_N(x, r_c, c, r_a, \mathbf{u})$ when there are no equality terminal constraints. This situation occurs, for instance, if there exists an $\varepsilon > 0$ such that $x_s(r) + \varepsilon \mathcal{B} \in \mathbb{X}_f(r)$ for all $r \in \mathcal{R}$.

Theorem 3.4: Suppose Assumptions 2.2-2.6 are satisfied. (i) Every trajectory of the controlled system $x^+ = f(x, \kappa_N(x, r_c, c))$ commencing in $\mathcal{X}_N^{\mathcal{R}}(\gamma)$ remains in $\mathcal{X}_N^{\mathcal{R}}(\gamma)$ and converges to $X^{\mathcal{R}}$ for all possible realizations of the reference sequence r . (ii) The set $X^{\mathcal{R}}$ is positively invariant for the controlled system $x^+ = f(x, \kappa_N(x, r_c, c))$ for all $r \in \mathcal{R}$ and $t \geq t_c \triangleq \min\{t > 0 \mid x(t) \in \mathcal{L}(r_c, \bar{c})\}$ where $\bar{c} = \max\{c^*(r^+), c^*(r_c)\}$.

Proof: - See the appendix.

We have shown that, if the reference trajectory takes arbitrary values in the compact set $\mathcal{R}$, then the state x converges in finite time to the set $X^{\mathcal{R}}$, a set that is large enough to guarantee good tracking properties. Also, if the reference output r is eventually equal to $r' \in \mathcal{R}$, then $\bar{x}(r')$ is asymptotically stable for the controlled system.

If the reference output r eventually lies in a set $\mathcal{R}' \subset \mathcal{R}$, it is possible to modify adaptively the precalculated constants $c^*(r)$ in order to guarantee that the state x converges to the smaller set $X^{\mathcal{R}'} \subset X^{\mathcal{R}}$.

Remark 2: Once the trajectory lies in $X^{\mathcal{R}}$, a disturbance acting on the system can move the state trajectory outside $X^{\mathcal{R}}$ and the optimization problem $\mathbb{P}_N(x, r_c, c)$ becomes infeasible. The occurrence of infeasibility in presence of disturbances can be prevented by converting the constraint $g_c(x, c, r_a, \mathbf{u}) \triangleq V_N(x, r_a, \mathbf{u}) - c \leq 0$ as a part of the cost function using exact penalty functions. The idea of exact penalty methods is to choose a penalty function $\psi(g_c(x, c, r_a, \mathbf{u})) : \mathbb{R} \rightarrow \mathbb{R}$ and a constant ρ sufficiently large (Polak 1997, Fletcher 2000) such that the solution of the modified optimization problem with the cost $J_N(x, r_c, c, r_a, \mathbf{u}) + \rho\psi(g_c(x, c, r_a, \mathbf{u}))$ is also the solution of the original problem $\mathbb{P}_N(x, r_c, c)$ under some regularity condition (Polak 1997, Fletcher 2000). This type of problems can be efficiently solved numerically using slack variables as follows.

$$\begin{aligned} \min_{\mathbf{u}, r_a} & J_N(x, r_c, c, r_a, \mathbf{u}) + \rho(|s_1| + |s_2|) \\ & g_c(x, c, r_a, \mathbf{u}) \leq s_1 - s_2, \quad s_1 \geq 0 \quad s_2 \geq 0 \\ & (\mathbf{u}, r_a) \in \mathcal{U}_N(x, c). \end{aligned}$$

A lower bound on the penalty parameter ρ guaranteeing the equivalence between the two optimization problems in the feasibility region of $\mathbb{P}_N(x, r_c, c)$ can be estimated (Polak 1997, Patel 2010, Rosenberg 1984).

4 Illustrative example

This example concerns the control of a continuous stirred tank reactor based on a model originally derived in Hicks and Ray (1971) and modified in Kameswaran and Biegler (2006). The system is described by

$$\begin{aligned}\dot{x}_1 &= (1/\theta)(1 - x_1) - kx_1e^{-(M/x_2)} \\ \dot{x}_2 &= (1/\theta)(x_f - x_2) + kx_1e^{-(M/x_2)} - \beta u(x_2 - x_c) \\ y &= x_2\end{aligned}$$

where x_1 is the product concentration, x_2 is the temperature and u is the coolant flow rate. The model parameters are $\theta = 20$, $k = 300$, $M = 5$, $x_f = 0.3947$, $x_c = 0.3816$ and $\beta = 0.117$. The state and control constraint sets are

$$\begin{aligned}\mathbb{X} &= \{x \in \mathbb{R}^2 \mid x_1 \in [0, 1], x_2 \in [0, 1]\}, \\ \mathbb{U} &= [0, 2],\end{aligned}$$

The discrete-time model is implicitly defined via the optimization process; the optimization packages described in Wächter and Biegler (2006), Wyk et al. (2010) were employed for all our simulations. Figure 2 shows the steady state temperature as a function of the coolant flow rate;

Figure 2.: Temperature steady state versus coolant flow rate

point A corresponds to $x(0) = [0.9831, 0.3918]$ while point B is the location of a Hopf bifurcation. The controller is required to track a time varying reference $r \in \mathcal{R} = [0.62, 0.68]$. The equilibrium pairs $(x_s(r), u_s(r))$ are as follow

$$\begin{aligned}x_{1,s}(r) &= \left(\frac{1}{1 + \theta k e^{-(M/r)}} \right), \quad x_{2,s}(r) = r \\ u_s(r) &= (1/\theta) \left(\frac{x_f + 1 - r - x_{1,s}(r)}{\beta u(r - x_c)} \right)\end{aligned}$$

and they are a set of a locally unstable steady states. The controller uses $N = 420$ with a prediction horizon $T_N = 420s$ corresponding to a sampling period of 1 second. The selected stage cost is $\ell(\tilde{x}(r_a), \tilde{u}(r_a)) = |x - x_s(r_a)|^2 + |u - u_s(r_a)|^2$. The terminal ingredients satisfying Assumption 2.2 have been chosen as $V_f(\tilde{x}(r_a), r_a) = 0$ and $\mathbb{X}_f(r_a) = \{x \mid x - x_s(r_a) = 0\}$ for simplicity. Finally $V_e(r_c - r_a) = L(r_c - r_a)^2$ where $L = 5000$. The large value for the penalty parameter L guarantees that the obtained optimal solution is close to the one given by the optimization problem with $r_a = r$ fixed, whenever it is feasible. For comparison, the model predictive controller for tracking proposed in Ferramosca et al. (2009), denoted as NL tracking MPC, is used with the same choice of the common ingredients. Figures 3 and 4 show the output

behavior of the controlled system for different time-varying reference signals. The simulations have been carried out using references originating critical situations but they are not a-priori known by the controller. In figure 3 the tracking of a square wave between 0.62 and 0.68 is illustrated.

Figure 3.: State variable x_2 vs time: reference trajectory (dotted), proposed algorithm (solid) and NL tracking MPC (dashed).

The reference used in figure 4 is sinusoidal with a frequency $\Omega = 0.02/3 \text{ rad/s}$. The simulation results show that NL tracking MPC can move the trajectories significantly far away from the region containing the equilibrium pairs when the reference is time varying. Contrarily the proposed scheme keeps the trajectories in a region characterized by $X^{\mathcal{R}}$ when the reference is time-varying. The tracking properties do not deteriorate with respect to the NL tracking MPC. The shape of $X^{\mathcal{R}}$ and the steady state error depends on Q and R . Indeed a proper tuning of the parameters Q and R might avoid the occurrence of the overshoot even in the NL tracking MPC but usually Q and R are chosen to impose simultaneously different types of performance and not just the overshoot. When the reference is randomly time-varying, the proposed algorithm guarantees that there will not be overshoot once Q and R are given.

Figure 4.: State variable x_2 vs time: reference trajectory (dotted), proposed algorithm (solid) and NL tracking MPC (dashed).

Figure 5, 6 and 7 compare the proposed algorithm and the NL tracking MPC when the stage cost is $\ell(\tilde{x}(r_a), \tilde{u}(r_a)) = |x - x_s(r_a)|_Q^2 + |u - u_s(r_a)|^2$ where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$ and the reference

range is $\mathcal{R} = [0.5, 0.7]$. In this case the range of references is large and consequently the sets X_r determined by $c^*(r)$ in equation (8) are wide as well. In the simulations the size of X_r has been reduced defining $X_r \triangleq \mathcal{L}(r, c^*(r)/20)$. The viability between equilibrium states is guaranteed by the fact that $x_s(r)$ is a monotonic Lipschitz continuous function in $\mathcal{R} = [0.5, 0.7]$. In figure 5 the tracking of a square wave between 0.5 and 0.7 is illustrated. Here we can see that the algorithm slowed the response to avoid overshoot when the references was changed abruptly. Figure 6 compares the inputs for the proposed algorithm and the NL tracking MPC. Generally

Figure 5.: State variable x_2 vs time: reference trajectory (dotted), proposed algorithm (solid) and NL tracking MPC (dashed).

the input variations of the proposed algorithm are more limited with respect to the NL tracking MPC. Figure 7 shows the obtained cost $V_N^0(x, r_c, c)$ for the NL tracking MPC and the proposed

Figure 6.: Input $u(t)$ vs time: proposed algorithm (solid) and NL tracking MPC (dashed).

approach. $V_N^0(x, r_c, c)$ obtained by the NL tracking MPC exhibits spikes for some of the changes in the reference.

5 Conclusion

It is well known, if the reference output for a system is constant or converging to a constant, that model predictive control may be used to ensure robust stability either of a state or a set of

Figure 7.: $V_N^0(x, r_c, c)$ vs time: proposed algorithm (solid) and NL tracking MPC (dashed); $c(t)$ (dotted).

states depending on the nature of the control problem. This article considers the case when the reference output ranges over a possibly large set and shows that, when this happens, the set of states to which the state converges is increased and that the region of attraction decreases. It is shown that these two sets can be quantified using a control Lyapunov function in the optimal control problem solved on-line.

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[Appendix: Proof of the results]

Proof of Proposition 3.1 - Let $z \triangleq (x, r_c, c)$, $\mathbb{P}_N(z)$ be feasible and $(\mathbf{u}^0(z), r_a^0(z))$ be the optimal solution. Then $\hat{r}_a(z^+) = r_a^0(z)$ and $\hat{\mathbf{u}}(z^+) = \{u^0(1; z), u^0(2; z), \dots, u^0(N-1; z), k_f(x^0(N; z), \hat{r}_a(z))\}$ are feasible solution to $\mathbb{P}_N(z^+)$ provided $k_f(\cdot, r_a^0(z))$ is a feedback law satisfying Assumption 2.2 and $x^0(N; z)$ is the state at time N if the initial state is x and the control input is $\mathbf{u}^0(z)$. The sequence $\hat{\mathbf{u}}(z^+)$ and $\hat{r}_a(z^+)$ are feasible at the next time instant because $\mathcal{U}_N(x^+, c^+)$ does not depends on r_c and c^+ given by equation (7) does not destroy feasibility.

Proof of Proposition 3.3 - Let $z \triangleq (x, r_c, c)$. We prove convergence of $r_a^0(t) \triangleq r_a^0(\varphi(t; z_0), r_c(t), c(t))$ and $\varphi(t; z_0)$ to $r(t)$ and $x_s(r(t))$ respectively for $r(t) = r$ constant, where $\varphi(t; z_0)$ with $z_0 = (x(0), r_c(0), c(0))$ is the solution of the closed loop system $x^+ = f(x, \kappa_N(z))$ and $\kappa_N(z)$ is defined in (3). Without loss of generality we impose $r_c = r$. Then

$$J_N^0(z^+) \leq J_N^0(z) - \ell(x - x_s(r_a^0(z)), \kappa_N(z) - u_s(r_a^0(z))) \quad (1)$$

since $\hat{r}_a(z^+) = r_a^0(z)$ and $J_N^0(z^+) \leq J_N(z^+, \hat{r}_a(z^+), \hat{\mathbf{u}}(z^+))$ where

$$\begin{aligned} J_N(z^+, \hat{r}_a(z^+), \hat{\mathbf{u}}(z^+)) &= V_N(x^+, r_a^0(z), \hat{\mathbf{u}}(z^+)) + V_e(r - r_a^0(z)) = \\ &= J_N^0(z) - \ell(x - x_s(r_a^0(z)), \kappa_N(z) - u_s(r_a^0(z))) + \\ &= V_f(\tilde{x}(N; x(1), r_a^0(z), \hat{\mathbf{u}}(z^+)), r_a^0(z)) - \\ &= V_f(\tilde{x}(N; x(0), r_a^0(z), \mathbf{u}^0(z)), r_a^0(z)) + \\ &= \ell(x(N-1; x(1), \hat{\mathbf{u}}(z^+)) - x_s(r_a^0(z)), k_f(x^0(N; z), r_a^0(z)) - u_s(r_a^0(z))) \end{aligned}$$

and by Assumption 2.2

$$V_f(\tilde{x}(N; x(1), r_a^0(z), \hat{\mathbf{u}}(z^+)), r_a^0(z)) - V_f(\tilde{x}(N; x(0), r_a^0(z), \mathbf{u}^0(z)), r_a^0(z)) + \ell(x(N-1; x(1), \hat{\mathbf{u}}(z^+)) - x_s(r_a^0(z)), k_f(x^0(N; z), r_a^0(z)) - u_s(r_a^0(z))) \leq 0.$$

From monotonicity of the value function established in (1) we have that $\lim_{t \rightarrow +\infty} J_N^0(z(t))$ exists and is non-negative. Then, due to the positiveness of the optimal cost we can infer that $\lim_{t \rightarrow +\infty} \varphi(t; z_0) - x_s(r_a^0(z(t))) = 0$, $\lim_{t \rightarrow +\infty} \kappa_N(z(t)) - u_s(r_a^0(z(t))) = 0$ and

$$\lim_{t \rightarrow +\infty} r_a^0(t) = \mathcal{R}_a \triangleq \{r_a \mid V_e(r - r_a) = \delta\}$$

where δ is a non-negative real value and, by Assumption 2.2 and relations (7), (8), there exists a $T > 0$ such that $V_N^0(z) \leq c^*(r)$ for all $t > T$. If the optimal solution is such that $r_a^0(t) = r(t) = r$ then convergence follows by standard arguments (Rawlings and Mayne 2009) by the property $V_N^0(z^+) - V_N^0(z) \leq \ell(x - x_s(r), \kappa_N(z) - u_s(r))$ where $V_N^0(z) \triangleq V_N(x, r_a^0(x, r, c^*(r)), \mathbf{u}^0(x, r, c^*(r)))$. Now consider the case when $r_a^0(t) \in \mathcal{R}_a$ for $\delta > 0$. We proceed by contradiction and show that $r_a^0(t)$ must converge to r and consequently the trajectory of the controlled system $x^+ = f(x, \kappa_N(z))$ converges to $x_s(r)$. The optimal solution $\mathbf{u}^0(z)$ of $\mathbb{P}_N(z)$ is, in general, a set valued map. We introduce the upper semi continuous map

$$\overline{\mathbf{u}}(z) \triangleq \mathbf{u}^0(z) \bigcup_i \{\hat{\mathbf{u}}_i\}$$

containing $\mathbf{u}^0(x, r, c^*(r))$, where $\hat{\mathbf{u}}_i$ are the limit points of every sequence $\{\mathbf{u}_i\}$ such that $\mathbf{u}_i \in \mathbf{u}^0(x_i, r, c^*(r))$ for every infinite sequence $\{x_i\}$ converging to x to close the graph. The limit points $\hat{\mathbf{u}}_i$ are feasible solutions since the set of admissible control $\mathcal{U}_N(x, c)$ is compact by continuity of $f(\cdot)$ and compactness of $\mathbb{U}$. The closed loop system satisfies

$$x^+ \in f(x, u^0(0; z)) \subseteq f(x, \overline{u}(0; z)) \quad (2)$$

where $f(x, \overline{u}(0; z))$ is an upper semi continuous function. In order to prove the convergence of the set of trajectories satisfying (2) we introduce the auxiliary function

$$\overline{V}_N(z) \triangleq \max_{\mathbf{u} \in \overline{\mathbf{u}}(z)} V_N(x, r_a^0(z), \mathbf{u}).$$

Then

$$\overline{V}_N(z^+) \leq \overline{V}_N(z) - \min_{u \in \overline{u}(0; z)} \ell(x - x_s(r_a^0(z)), u - u_s(r_a^0(z))) \quad (3)$$

since

$$\max_{u \in u^0(1; z)} \ell(x^+ - x_s(r_a^0(z^+)), u - u_s(r_a^0(z^+))) \geq \max_{u \in \overline{u}(0; z^+)} \ell(x^+ - x_s(r_a^0(z^+)), u - u_s(r_a^0(z^+))).$$

Hence, the upper semi continuity of the set valued map $f(x, \overline{u}(0; z))$ guarantees that the set of all possible solutions $\varphi(t; z_0)$ starting from z_0 and satisfying $\varphi(t+1; z_0) \in f(\varphi(t; z_0), \overline{u}(0; \varphi(t; z_0), r, c))$ converges to a weakly invariant omega limit set $\Omega(x, r) \triangleq x_s(\mathcal{R}_a)$. Notice that $\Omega(x, r)$ is only made up of equilibrium pairs as a consequence of the decreasing condition (3) on the auxiliary function. Pick $x_s(r_a) \in \Omega(x, r)$ as the initial condition for the problem $\mathbb{P}_N(\cdot)$ with $u^0(t; z) = u_s(r_a)$ for all $t > 0$ so that $V_N(x_s(r_a), r_a, \mathbf{u}^0) = 0$. It follows that $V_e(r - r_a) > 0$. Since by Assumptions 2.2, 2.3, 2.5, 2.4 and 2.6 there exist

positive constants $k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7$ and k_8 such that $|V_f(\tilde{x}(N; x_s(r_a), r_a, \mathbf{u}^0), r_a) - V_f(\tilde{x}(N; x_s(r_a), r'_a, \mathbf{u}'), r'_a)| = |V_f(\tilde{x}(N; x_s(r_a), r'_a, \mathbf{u}'), r'_a)| \leq k_0(|\tilde{x}(N; x_s(r_a), r'_a, \mathbf{u}')|^2) = k_0(|x(N; x_s(r_a), \mathbf{u}') - x_s(r'_a)|^2) \leq k_1(|x_s(r_a) - x_s(r'_a)|^2) + k_2(|x(N; x_s(r_a), \mathbf{u}') - x_s(r_a)|^2) = k_1(|x_s(r_a) - x_s(r'_a)|^2) + k_2(|x(N; x_s(r_a), \mathbf{u}') - x(N; x_s(r_a), \mathbf{u}^0)|^2) \leq k_4|r_a - r'_a|^2 + k_5(|\mathbf{u}' - \mathbf{u}^0|^2)$ and similarly $|\ell(\tilde{x}(i; x_s(r_a), r_a, \mathbf{u}^0), \tilde{u}(i; x_s(r_a), r_a, \mathbf{u}^0)) - \ell(\tilde{x}(i; x_s(r_a), r'_a, \mathbf{u}'), \tilde{u}(i; x_s(r_a), r'_a, \mathbf{u}'))| = |\ell(\tilde{x}(i; x_s(r_a), r'_a, \mathbf{u}'), \tilde{u}(i; x_s(r_a), r'_a, \mathbf{u}'))| \leq k_6|r_a - r'_a|^2 + k_7(|u'(i) - u_s(r_a)|^2) + k_8(|\mathbf{u}' - \mathbf{u}^0|^2)$ the variation of the cost $J_N(z, r_a, \mathbf{u}^0)$ with respect to perturbations in r_a is exclusively determined by $V_e(r - r_a)$ since there exists, by Assumption 2.1, a feasible $\mathbf{u}'$ such that $\mathbf{u}' \rightarrow \mathbf{u}^0$ as $r'_a \rightarrow r_a$. Then, by Assumptions 2.6 and 2.4, there exist feasible $\hat{r} \notin \mathcal{R}_a$, $\hat{\mathbf{u}}$ such that $J_N(z, \hat{r}, \hat{\mathbf{u}}) < V_e(r - r_a)$.

We conclude that $\Omega(x, r)$ can not be a weakly invariant omega limit set. Thus $r_a^0(t)$ converges to r and $\varphi(t; z_0)$ to $x_s(r)$.

Prof. of theorem 3.4 - (i) The convergence of the closed loop trajectory $x(t)$ to $X^{\mathcal{R}}$ in finite time is guaranteed by maintaining r_c constant for $V_N(x, r_a^0(z), \mathbf{u}^0(z)) \geq c^*(r(t))$ and it is granted by Proposition 3.3. The virtual reference r_c can only change when the state trajectory lies in $X^{\mathcal{R}}$. Since $\mathcal{X}_N^{\mathcal{R}}(\gamma)$ describes the feasibility region of the optimization problem $\mathbb{P}_N(\cdot)$ at time 0, the positive invariance is guaranteed by feasibility at any time, once the feasibility at the first time instant is obtained since $\mathcal{X}_N^{\mathcal{R}}(c) \subseteq \mathcal{X}_N^{\mathcal{R}}(\gamma)$. (ii) The positive invariance of $X^{\mathcal{R}}$ is guaranteed by the constraints $V_N(x, r_a, \mathbf{u}) \leq c \forall r_a \in \mathcal{R}$ for all $r \in \mathcal{R}$ and $t \geq t_c$ where $t_c \triangleq \min\{t > 0 \mid x(t) \in \mathcal{L}(r_c, \bar{c})\}$ where $\bar{c} = \max\{c^*(r^+), c^*(r_c)\}$ and the selection policy of r_c^+ which uses the previous reference whenever $V_N(x(t), r_a^0(z(t)), \mathbf{u}^0(z(t))) \geq c^*(r(t) + \xi)$ allowing the decreasing of $V_N(x, r_a, \mathbf{u})$ inside $X^{\mathcal{R}}$.

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