SUB-GAUSSIAN ESTIMATES OF HEAT KERNELS ON INFINITE GRAPHS

ALEXANDER GRIGOR'YAN AND ANDRAS TELCS

Abstract

We prove that a two-sided sub-Gaussian estimate of the heat kernel on an infinite weighted graph takes place if and only if the volume growth of the graph is uniformly polynomial and the Green kernel admits a uniform polynomial decay.

Contents

1. Introduction	452
2. Statement of the main result	455
3. Preliminaries	459
4. Outline of the proof and its consequences	464
5. The Faber-Krahn inequality and on-diagonal upper bounds	468
6. The mean exit time and the Green kernel	476
7. Sub-Gaussian term	478
8. Off-diagonal upper bound of the heat kernel	482
9. On-diagonal lower bound	484
10. The Harnack inequality and the Green kernel	485
11. Oscillation inequalities	488
12. Time derivative of the heat kernel	490
13. Off-diagonal lower bound	493
14. Parity matters	499
15. Consequences of the heat kernel estimates	503
Appendix. The list of the lettered conditions	505
References	506

1. Introduction

Consider the heat equation

$$\frac{\partial f}{\partial t} = \Delta f, \quad (1.1)$$

where $f = f(t, x)$ is a function of $t > 0$ and $x \in \mathbb{R}^n$, and where Δ is the Laplace operator in $\mathbb{R}^n$. The fundamental solution to (1.1) is given by the classical Gauss-Weierstrass formula

$$f(t, x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

The function $p_t(x, y) = f(t, x - y)$ is called the *heat kernel* of the Laplace operator.

In the past three decades, there have been numerous works devoted to estimates of heat kernels in various settings (see, e.g., books and surveys [4], [15], [16], [25], [35], [52], [55], [65], [66]). These are parabolic equations with variable coefficients, the heat equation on Riemannian manifolds, the discrete heat equation on graphs, and the heat semigroups on general metric measure spaces including fractal-like sets. Despite the high diversity of the underlying spaces and equations, in many important cases the heat kernel is naturally defined and, moreover, admits the so-called Gaussian estimates.

For any metric measure space M with distance d and measure μ , denote by $B(x, r)$ the open metric ball of radius r centered at x , and denote by $V(x, r)$ its measure μ . Suppose first that M is either a discrete group or a Lie group, with properly defined d, μ and the heat kernel $p_t(x, y)$. Assume that the volume growth of M is polynomial; that is, for some $\alpha > 0$,

$$V(x, r) \simeq r^\alpha. \quad (1.2)$$

(Here the sign $\simeq$ means that the ratio of both sides of (1.2) stays between two positive constants.) Then the heat kernel on M admits the following Gaussian estimate (see [64], [37]):

$$p_t(x, y) \simeq t^{-\alpha/2} \exp\left(-\frac{d^2(x, y)}{ct}\right) \quad (1.3)$$

(where the positive constant c may be different for the upper and lower bounds). The heat kernel in $\mathbb{R}^n$ obviously satisfies (1.3) with $\alpha = n$.

Suppose now that M is a complete manifold with nonnegative Ricci curvature. Then the following estimate of P. Li and S.-T. Yau [47] is well known:

$$p_t(x, y) \simeq \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct}\right). \quad (1.4)$$

In particular, if $V(x, r) \simeq r^\alpha$, then the heat kernel again satisfies the estimate (1.3).

As we see, for groups of polynomial growth and for nonnegatively curved manifolds, the heat kernel is fully determined (up to constant factors) by the volume growth function. In other words, the potential theory on such spaces is characterized by a single parameter α —the exponent of the volume growth.

The presence of Gaussian estimates (1.3) or (1.4) reflects certain properties of the space M . In particular, (1.4) implies that the Markov process X_t with the transition density $p_t(x, y)$ has the diffusion speed of the order $t^{1/2}$. The latter means that the process X_t , started at a point x , first exits the ball $B(x, R)$ at the time $t \simeq R^2$.

The development of Markov processes on fractals and fractal-like graphs (see [10], [7], [30], [36], [40], [41], [42], [44], [45], [59], [67], etc.) has led to construction of homogeneous metric spaces M where the process X_t has the diffusion speed of the order $t^{1/\beta}$, with some $\beta > 2$. Such a process X_t is referred to as *subdiffusive*, and it is characterized by two parameters α and β , which determine *sub-Gaussian estimates* of the heat kernel:

$$p_t(x, y) \simeq t^{-\alpha/\beta} \exp \left(- \left(\frac{d^\beta(x, y)}{ct} \right)^{1/(\beta-1)} \right). \quad (1.5)$$

Here α is the exponent of the volume growth as in (1.2). The Gaussian estimate (1.3) is a particular case of (1.5) for $\beta = 2$.

M. Barlow and R. Bass [7] showed that sub-Gaussian estimates (1.5) with $\beta > 2$ can take place not only on singular spaces such as fractals but also on smooth Riemannian manifolds, for a certain range of time. Similar estimates hold for random walks on certain fractal-like graphs (see [8], [39]). It has become apparent that a large and interesting class of homogeneous spaces features sub-Gaussian estimates of the heat kernel. The potential theory on such spaces is determined by the two parameters and hence cannot be recovered only from the volume growth.*

A natural question arises:

How do we characterize those spaces that admit sub-Gaussian estimates (1.5) of the heat kernel?

If M is a complete noncompact Riemannian manifold, then the validity of Gaussian estimate (1.3) is known to be equivalent to the following two conditions: volume growth (1.2) and the Poincaré inequality

$$\lambda_1^{(N)}(B(x, r)) \geq \frac{c}{r^2}, \quad (1.6)$$

where $\lambda_1^{(N)}(B)$ is the first nonzero eigenvalue of the Neumann boundary value problem in the ball B (see [53], [31]; similar results are known also for graphs (see [28])

*The parameters α and β must satisfy the inequalities $2 \leq \beta \leq \alpha + 1$, which seem to be the *only* constraints on α and β . We are indebted to Martin Barlow for providing us with the evidence for the latter.

and for abstract local Dirichlet spaces (see [57])). It may be tempting to conjecture that by replacing r^2 by r^β in (1.6), one obtains equivalent conditions for sub-Gaussian estimates. However, this conjecture is false. At the present time, no similar characterization of spaces with sub-Gaussian estimates seems to be known. All examples of spaces where (1.5) is proved are fractal-like spaces featuring a self-similarity structure.

The purpose of this paper is to provide a new approach to obtaining sub-Gaussian estimates of the heat kernel. Our point of departure is the understanding that, apart from the uniform volume growth $V(x, r) \simeq r^\alpha$, we have to introduce additional hypotheses, which would contain the second parameter β and provide the necessary homogeneity of the space. (Just the uniform volume growth is not enough for the latter.)

Let $g(x, y)$ be the Green kernel on M ; that is,

$$g(x, y) = \int_0^\infty p_t(x, y) dt.$$

Recall that, in $\mathbb{R}^n$, $g(x, y) = c_n |x - y|^{-(n-2)}$ if $n > 2$ and $g \equiv \infty$ if $n \leq 2$.

Our general result says the following:

Given the parameters $\alpha > \beta \geq 2$, the two-sided sub-Gaussian estimate

$$p_t(x, y) \simeq t^{-\alpha/\beta} \exp \left(- \left(\frac{d^\beta(x, y)}{ct} \right)^{1/(\beta-1)} \right) \quad (1.7)$$

holds if and only if

$$V(x, r) \simeq r^\alpha \quad \text{and} \quad g(x, y) \simeq d(x, y)^{-(\alpha-\beta)}. \quad (1.8)$$

We do not specify here the ranges of the variables x, y, t, r because they are different for different settings. In the present paper, we treat the case when the underlying space is a graph, and the time is also discrete. However, the graph case already contains all difficulties. We present the proof in such a way that only minimal changes are required to pass to a general setting of abstract metric spaces, which will be dealt with elsewhere. The exact statements are given in Section 2. Note that our result is new, even for the Gaussian case $\beta = 2$.

Hypothesis (1.8) consists of two conditions of different nature. The first one is a geometric condition of the volume growth, whereas the second is an estimate of a fundamental solution to an elliptic equation. Neither of them separately implies heat kernel bounds (1.7). Surprisingly enough, the exponent β , which provides the scaling of space and time variables for a *parabolic* equation, can be recovered from an *elliptic* equation, although combined with the volume growth.

The paper is arranged as follows. In Section 2 we state the main result—Theorem 2.1. In Section 3 we introduce necessary tools such as the discrete Laplace operator,

its eigenvalues, the mean exit time, and so on. In Section 4 we describe the scheme of the proof of Theorem 2.1, as well as some consequences. In particular, we mention some other conditions equivalent to (1.7). The actual proof of Theorem 2.1 consists of many steps that are considered in detail in Sections 5–15.

Notation. The letters c, C are reserved for positive constants not depending on the variables in question. They may be different on different occurrences, even within the same formula. All results of the paper are quantitative in the sense that the constants in conclusions depend only on the constants in hypotheses.

The relation $f \simeq g$ means that the ratio of functions f and g is bounded from above and below by positive constants, for the specified range of the variables. If one of those functions contains a sub-Gaussian factor $\exp(-(d^\beta/(ct))^{1/(\beta-1)})$, then the constant c in exp may be different for the upper and lower bounds (cf. (1.7)).

We use a number of lettered formulas such as (UE) , (LE) , and so on, to refer to the most important and frequently used conditions. In the appendix we provide a complete list of all such formulas.

2. Statement of the main result

Throughout the paper, Γ denotes an infinite, connected, locally finite graph. If $x, y \in \Gamma$, then we write $x \sim y$, provided x and y are connected by an edge. The graph is always assumed nonoriented; that is, $x \sim y$ is equivalent to $y \sim x$. We do not exclude loops so that $x \sim x$ is possible. If $x \sim y$, then $\overline{xy}$ denotes the edge connecting x and y . The distance $d(x, y)$ is the minimal number of edges in any edge path connecting x and y .

Assume that graph Γ is endowed by a *weight* μ_{xy} , which is a symmetric nonnegative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if $x \sim y$. Given μ_{xy} , we also define a measure μ on vertices by

$$\mu(x) := \sum_{y \sim x} \mu_{xy}$$

and

$$\mu(A) := \sum_{x \in A} \mu(x),$$

for any finite set $A \subset \Gamma$. The couple (Γ, μ) is called a *weighted graph*. Here μ refers both to the weight μ_{xy} and to the measure μ .

Any graph Γ admits a *standard weight*, which is defined by $\mu_{xy} = 1$ for all edges $\overline{xy}$. For such a weight, $\mu(x)$ is equal to the degree of the vertex x , which is the number of its neighbors.

Any weighted graph has a natural *Markov operator* $P(x, y)$ defined by

$$P(x, y) := \frac{\mu_{xy}}{\mu(x)}. \quad (2.1)$$

Clearly, we have

$$\sum_{y \in \Gamma} P(x, y) = 1 \quad (2.2)$$

and

$$P(x, y)\mu(x) = P(y, x)\mu(y). \quad (2.3)$$

For the Markov operator P , there is an associated random walk X_n , jumping at each time $n \in \mathbb{N}$ from a current vertex x to a neighboring vertex y with probability $P(x, y)$. The process X_n is Markov and reversible with respect to measure μ . If μ is the standard weight on Γ , then X_n is called a *simple random walk* on Γ .

Conversely, given a countable set Γ with a measure μ and a Markov operator $P(x, y)$ on Γ satisfying (2.3), identity (2.1) uniquely determines a symmetric weight μ_{xy} on $\Gamma \times \Gamma$. Then one defines edges $\bar{x}\bar{y}$ as those pairs of vertices for which $\mu_{xy} \neq 0$, and one obtains a weighted graph (Γ, μ) . One has to assume in addition that the resulting graph Γ is connected and locally finite.

Let P_n denote the n th convolution power of the operator P . Alternatively, $P_n(x, y)$ is the transition function of the random walk X_n ; that is,

$$P_n(x, y) = \mathbb{P}_x(X_n = y).$$

Define also the transition density of X_n , or the *heat kernel*, by

$$p_n(x, y) := \frac{P_n(x, y)}{\mu(y)}.$$

As obviously follows from (2.3), $p_n(x, y) = p_n(y, x)$.

The only a priori assumption that we normally make about the transition probability is the following:

$$P(x, y) \geq p_0, \quad \forall x \sim y, \quad (p_0)$$

where p_0 is a positive constant. Due to (2.2), hypothesis (p_0) implies that the degree of each vertex $x \in \Gamma$ is uniformly bounded from above. The latter is in fact equivalent to (p_0) , provided X_n is a simple random walk.

By *sub-Gaussian* heat kernel estimates on graphs, we mean the following inequalities:

$$p_n(x, y) \leq Cn^{-\alpha/\beta} \exp\left(-\left(\frac{d(x, y)^\beta}{Cn}\right)^{1/(\beta-1)}\right) \quad (UE)$$

and

$$p_n(x, y) + p_{n+1}(x, y) \geq cn^{-\alpha/\beta} \exp\left(-\left(\frac{d(x, y)^\beta}{cn}\right)^{1/(\beta-1)}\right), \quad n \geq d(x, y), \quad (LE)$$

where x, y are arbitrary points on Γ and where n is a positive integer.

Let us comment on the differences between (UE) and (LE) . First, observe that $p_n(x, y) = 0$ whenever $n < d(x, y)$. (Indeed, the random walk cannot get from x to y in a number of steps smaller than $d(x, y)$.) Therefore, the restriction $n \geq d(x, y)$ in (LE) is necessary. We could assume the same restriction in (UE) , but if $p_n(x, y) = 0$, then (UE) is true anyway. Another difference—using $p_n + p_{n+1}$ in (LE) in place of p_n in (UE) —is due to the parity problem. Indeed, if graph Γ is bipartite (e.g., $\mathbb{Z}^D$), then $p_n(x, y) = 0$ whenever n and $d(x, y)$ have different parities. Therefore, the lower bound for p_n cannot hold in general, and we state it for $p_n + p_{n+1}$ instead. Alternatively, one could say that the lower bound holds either for p_n or for p_{n+1} . The structure of the graph may cause one of p_n, p_{n+1} to be small (or even vanish), but it is not possible to decide a priori which of these two terms admits the lower bound (see Section 14 for more details).

Denote by $B(x, R)$ a ball on Γ of radius R centered at x , and denote by $V(x, R)$ its measure, that is,

$$B(x, R) := \{y \in \Gamma : d(x, y) < R\}, \quad V(x, R) := \mu(B(x, R)).$$

We say that graph (Γ, μ) has the *regular volume growth* of degree α if

$$V(x, R) \simeq R^\alpha, \quad \forall x \in \Gamma, R \geq 1. \quad (V)$$

The *Green kernel* of (Γ, μ) is defined by

$$g(x, y) := \sum_{n=0}^{\infty} p_n(x, y).$$

Assuming that $\alpha > \beta$, the estimates (UE) and (LE) imply, upon summation in n ,

$$g(x, y) \simeq d(x, y)^{-\gamma}, \quad \forall x \neq y, \quad (G)$$

where $\gamma = \alpha - \beta$. It turns out that (G) , together with the volume growth condition (V) , is sufficient to recover the heat kernel estimates (UE) and (LE) , as is stated in the following main theorem.

THEOREM 2.1

Let $\alpha > \beta > 1$, and let $\gamma = \alpha - \beta$. For any infinite connected weighted graph (Γ, μ) satisfying (p_0) , the following equivalence holds:

$$(V) + (G) \iff (UE) + (LE).$$

Remark 2.1

Under hypotheses (V) and (G), some partial heat kernel estimates were obtained by A. Telcs [62].

It is well known that a simple random walk in $\mathbb{Z}^D$ admits the Gaussian estimate

$$cn^{-D/2} \exp\left(-\frac{d^2(x, y)}{cn}\right) \leq p_n(x, y) \leq Cn^{-D/2} \exp\left(-\frac{d^2(x, y)}{Cn}\right), \quad (2.4)$$

subject to the restrictions $n \equiv d(x, y) \pmod{2}$ and $d(x, y) \leq n$. Similar Gaussian estimates were also proved for more general graphs, under various assumptions (see [37], [54], [22], [28]). It is easy to see that (2.4) is equivalent to $(UE) + (LE)$ for $\alpha = D$ and $\beta = 2$ (see Section 14 for the parity matters).

Barlow and Bass [8] constructed a family of graphs—graphical Sierpiński carpets (resembling in the large scale the multi-dimensional Sierpiński carpet), which are characterized by the parameters α and β . Heat kernels on those graphs satisfy sub-Gaussian estimates (UE) and (LE) . In general, the parameters α and β in (UE) and (LE) must satisfy the inequalities

$$2 \leq \beta \leq \alpha + 1, \quad (2.5)$$

which can be seen as follows. By [9, Th. 2.1], the lower bound in (V) implies the on-diagonal upper bound $p_n(x, x) \leq Cn^{-\alpha/(\alpha+1)}$. By the result of [48], the upper bound in (V) implies the on-diagonal lower bound $p_n(x, x) \geq c(n \log n)^{-\alpha/2}$. Comparing these estimates with the on-diagonal lower and upper bounds implied by (LE) and (UE) , we obtain (2.5) (cf. [4, Th. 3.20 and Rem. 3.22], [59], as well as Lemma 5.4).

The sub-Gaussian estimates for different α and β are related as follows. Consider the right-hand side of (UE) and (LE) as a function of α and β . It is easy to see that it decreases as β and α/β simultaneously increase (assuming $d(x, y) \geq n$). In particular, (UE) gets stronger (and (LE) gets weaker) on increasing of α with constant β , whereas in general there is no monotonicity in β .

Estimates (UE) and (LE) were proved by O. Jones [39] for the graphical Sierpiński gasket. The latter is a graph that is obtained from an equilateral triangle by a fractal-like construction (see Figure 1). The reason for a subdiffusive behaviour of the random walk on such graphs is that they contain plenty of “holes” of all sizes, which causes the random walk to spend more time on circumventing the obstacles rather than on moving away from the origin.

It is possible to show that (V) and (G) imply $\beta \geq 2$ (see Lemma 5.4). The assumption $\alpha > \beta$ is necessary to ensure the finiteness of the Green function. It is known that either $g(x, y)$ is finite for all x, y or $g \equiv \infty$. In the first case the graph

Figure 1. A fragment of the graphical Sierpiński gasket

(Γ, μ) is called *transient* and in the second case *recurrent* (e.g., $\mathbb{Z}^D$ is transient if $D \geq 3$ and recurrent otherwise). Hence, Theorem 2.1 serves only transient graphs.

The question of finding equivalent conditions for sub-Gaussian estimates (UE) and (LE) is equally interesting for recurrent graphs. Note that the graph in Figure 1 is recurrent.* Indeed, the volume function on this graph obviously admits the estimate

$$V(x, r) \leq Cr^2,$$

which implies the recurrence (see [18], [66]). Alternatively, one can see directly that $\alpha < \beta$ because the parameters α and β for the Sierpiński gasket are $\alpha = \log 3 / \log 2$ and $\beta = \log 5 / \log 2$ (see [4]).

Some hints on the recurrent case are given in Section 4.

3. Preliminaries

If P is the Markov operator of a weighted graph (Γ, μ) and if I is the identity operator, then $\Delta := P - I$ is called the *Laplace operator* of (Γ, μ) . For any set $A \subset \Gamma$, denote by $\overline{A}$ the set containing all vertices of A and all their neighbors. If a function f is defined on $\overline{A}$, then Δf is defined on A and

$$\Delta f(x) = \sum_{y \sim x} P(x, y) f(y) - f(x) = \frac{1}{\mu(x)} \sum_{y \in \Gamma} (\nabla_{xy} f) \mu_{xy}, \tag{3.1}$$

*Plenty of examples of transient graphs and fractals with sub-Gaussian heat kernel bounds can be found in [4], [7], and [8].

where

$$\nabla_{xy} f := f(y) - f(x).$$

Note that although the summation in the second sum in (3.1) runs over all vertices y , the summand is nonvanishing only if $y \sim x$.

The following is a discrete analogue of the Green formula: for any finite set A and for all functions f and g defined on $\bar{A}$,

$$\sum_{x \in A} \Delta f(x) g(x) \mu(x) = \sum_{x \in A, y \notin A} (\nabla_{xy} f) g(x) \mu_{xy} - \frac{1}{2} \sum_{x, y \in A} (\nabla_{xy} f) (\nabla_{xy} g) \mu_{xy}. \quad (3.2)$$

We say that a function v is *harmonic* in set A if v is defined in $\bar{A}$ and $\Delta v = 0$ in A . Similarly, we say that a function v is *superharmonic* if $\Delta v \leq 0$. Observe that the inequality $\Delta v \leq 0$ is equivalent to

$$v(x) \geq \sum_{y \sim x} P(x, y) v(y).$$

The latter implies, in particular, that the infimum of a family of superharmonic functions is again superharmonic.

For any nonempty set $A \subset \Gamma$, let $c_0(A)$ be the set of functions on Γ whose support is finite and is in A . Denote by Δ_A the Laplace operator with the vanishing Dirichlet boundary condition on A ; that is,

$$\Delta_A f(x) := \begin{cases} \Delta f, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The operator Δ_A is symmetric with respect to the measure μ and is nonpositive definite. Moreover, it is essentially self-adjoint in $L^2(A, \mu)$.

For a finite set A , denote by $|A|$ its cardinality. If A is finite and nonempty, then the operator $-\Delta_A$ has $|A|$ nonnegative eigenvalues that we enumerate in increasing order and denote as follows:

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_{|A|}(A).$$

It is known that all eigenvalues $\lambda_i(A)$ lie in the interval $[0, 2]$ and that $\lambda_1(A) \in [0, 1]$ (see, e.g., [19], [22, Sec. 3.3]). The smallest eigenvalue $\lambda_1(A)$ admits the variational definition

$$\lambda_1(A) = \inf_{f \in c_0(A)} \frac{-(\Delta f, f)}{(f, f)} = \inf_{f \in c_0(A)} \frac{(1/2) \sum_{x \sim y} (\nabla_{xy} f)^2 \mu_{xy}}{\sum_x f^2(x) \mu(x)}, \quad (3.3)$$

where

$$(f, g) := \sum_{x \in \Gamma} f(x) g(x) \mu(x).$$

If $A = B(x, R)$, then we write, for simplicity,

$$\lambda(x, R) := \lambda_1(B(x, R)).$$

Given a nonempty set $A \subset \Gamma$, let X_n^A be the random walk on (Γ, μ) with the killing condition outside A . Its Markov operator $P^A(x, y)$ is defined by

$$P^A(x, y) := \begin{cases} P(x, y), & x, y \in A, \\ 0, & \text{otherwise.} \end{cases}$$

The transition function $P_n^A(x, y)$ of X_n^A is defined inductively: $P_0^A(x, y) = \delta_{xy}$ and

$$P_{n+1}^A(x, y) = \sum_{z \in \Gamma} P_n^A(x, z) P^A(z, y) = \sum_{z \in \Gamma} P^A(x, z) P_n^A(z, y). \quad (3.4)$$

As easily follows from (3.4), the function $u_n(x) = P_n^A(x, y)$ satisfies in $A \times \mathbb{N}$ the discrete heat equation

$$u_{n+1} - u_n = \Delta_A u_n. \quad (3.5)$$

The heat kernel $p_n^A(x, y)$ of X_n^A is defined by

$$p_n^A(x, y) := \frac{P_n^A(x, y)}{\mu(y)}.$$

As follows from (2.1), p^A is symmetric in x and y . In particular, the kernel $p_n^A(x, y)$ satisfies heat equation (3.5) both in (n, x) and (n, y) . If $f(x)$ is a function on A , then the function

$$u_n(x) := P_n^A f(x) = \sum_{y \in A} p_n^A(x, y) f(y) \mu(y)$$

solves, in $A \times \mathbb{N}$, heat equation (3.5) with initial data $u_0 = f$ and boundary data $u_n(x) = 0$ if $x \notin A$.

The Green function of X_n^A is defined by

$$G_A(x, y) := \sum_{n=0}^{\infty} P_n^A(x, y).$$

The alternative definition is that the function $G_A(x, y)$ is the infimum of all positive fundamental solutions of the Laplace equation in A . If the Green function is finite, then, for any $y \in A$, we have $\Delta_A G_A(\cdot, y) = -\delta_y$. The opposite case, when $G_A(x, y) \equiv +\infty$, is equivalent to the recurrence of the process X_n^A .

The Green kernel $g_A(x, y)$ is defined by

$$g_A(x, y) = \frac{G_A(x, y)}{\mu(y)} = \sum_{n=0}^{\infty} p_n^A(x, y).$$

Clearly, the Green kernel is symmetric in x, y . Therefore, if g_A is finite, then g_A is superharmonic in A with respect to both x and y , and it is harmonic away from the diagonal $x = y$. Observe that if $\mu(x) \simeq 1$ (which in particular follows from (V)), then $G_A(x, y) \simeq g_A(x, y)$ and $p_n^A(x, y) \simeq P_n^A(x, y)$.

It is easy to see that kernels $p_n^A(x, y)$ and $g_A(x, y)$ increase when A is enlarged and tend to global kernels $p_n(x, y)$ and $g(x, y)$ (defined in Section 2) as an increasing sequence of sets A exhausts Γ .

If A is finite and nonempty, then it makes sense to consider the Dirichlet problem in A ,

$$\begin{cases} \Delta u = f & \text{in } A, \\ u = h & \text{in } \bar{A} \setminus A, \end{cases} \quad (3.6)$$

where f and h are given function on A and $\bar{A} \setminus A$, respectively. As follows easily from the maximum principle, the solution u exists and is unique. For a finite set A , $c_0(A)$ is identified with all functions on A extended by zero outside A . Then the equation

$$\Delta_A u = f,$$

where u and f are in $c_0(A)$, is equivalent to Dirichlet problem (3.6) with $h = 0$. Its solution is given by means of the Green operator G_A as follows:

$$u(x) = -G_A f(x) = -\sum_y G_A(x, y) f(y). \quad (3.7)$$

In other words, we have $G_A = (-\Delta_A)^{-1}$.

For any set $A \subset \Gamma$ and a point $x \in \Gamma$, define the *mean exit time* $E_A(x)$ by

$$E_A(x) := \sum_{y \in A} G_A(x, y). \quad (3.8)$$

As follows from the above discussion, the function $E_A(x)$ solves the following boundary value problem in A :

$$\begin{cases} \Delta u = -1 & \text{in } A, \\ u = 0 & \text{outside } A. \end{cases} \quad (3.9)$$

Denote by T_A the first exit time from set A for the process X_n ; that is,

$$T_A := \min\{k : X_k \notin A\}.$$

We claim that $E_A(x) = \mathbb{E}_x(T_A)$, which justifies the term “mean exit time” for E_A . Indeed, T_A coincides with the cardinality of all $n = 0, 1, 2, \dots$ for which X_n^A is in A ; that is,

$$T_A = \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n^A \in A\}},$$

whence

$$\mathbb{E}_x(T_A) = \sum_{n=0}^{\infty} \mathbb{P}_x \left(X_n^A \in A \right) = \sum_{n=0}^{\infty} \sum_{y \in A} P_n^A(x, y) = \sum_{y \in A} G_A(x, y) = E_A(x).$$

If $A = B(x, R)$, then we use a shorter notation

$$E(x, R) := E_{B(x, R)}(x).$$

Another function associated with the exit time is the *exit probability*, defined by

$$\Psi_n^A(x) := \mathbb{P}_x \{X_k \notin A \text{ for some } k \leq n\} = \mathbb{P}_x \{T_A \leq n\}. \quad (3.10)$$

In other words, $\Psi_n^A(x)$ is the probability that the random walk X_k started at x will at least once exit A by time n . Alternatively, $\Psi_n^A(x)$ can be defined as the solution $u_n(x)$ to the following initial boundary value problem in $A \times \mathbb{N}$:

$$\begin{cases} u_{n+1} - u_n = \Delta u_n, \\ u_0(x) = 0, & x \in A, \\ u_n(x) = 1, & x \notin A \text{ and } n \geq 0. \end{cases} \quad (3.11)$$

If $A = B(x, R)$, then we use the shorter notation

$$\Psi_n(x, R) := \Psi_n^{B(x, R)}(x).$$

To conclude this section, we prove two useful consequences of condition (p_0) :

$$P(x, y) \geq p_0, \quad \forall x \sim y. \quad (p_0)$$

PROPOSITION 3.1

If (p_0) holds, then, for all $x \in \Gamma$ and $R > 0$ and for some $C = C(p_0)$,

$$V(x, R) \leq C^R \mu(x). \quad (3.12)$$

Remark 3.1

Inequality (3.12) implies that, for a bounded range of R , $V(x, R) \simeq \mu(x)$.

Proof

Let $x \sim y$. Since $P(x, y) = \mu_{xy}/\mu(x)$ and $\mu_{xy} \leq \mu(y)$, hypothesis (p_0) implies $p_0\mu(x) \leq \mu(y)$. Similarly, $p_0\mu(y) \leq \mu(x)$. Iterating these inequalities, we obtain, for arbitrary x and y ,

$$p_0^{d(x, y)} \mu(y) \leq \mu(x). \quad (3.13)$$

Another consequence of (p_0) is that any point x has at most p_0^{-1} neighbors. Therefore, any ball $B(x, R)$ has at most C^R vertices inside. By (3.13), any point $y \in B(x, R)$ has measure at most $p_0^{-R} \mu(x)$, whence (3.12) follows. $\square$

PROPOSITION 3.2

Assume that hypothesis (p_0) holds on (Γ, μ) . Let function v be nonnegative in $\overline{A}$ and superharmonic in A . Then, for all points $x, y \in A$ such that $x \sim y$, we have $v(x) \simeq v(y)$.

Proof

Indeed, the superharmonicity of v implies

$$v(x) \geq \sum_{z \sim x} P(x, z) v(z) \geq P(x, y) v(y),$$

whence $v(x) \geq p_0 v(y)$ by (p_0) . In the same way, $v(y) \geq p_0 v(x)$, whence the claim follows. $\square$

4. Outline of the proof and its consequences

The proof of Theorem 2.1 consists of many steps. Here we describe the logical order of these steps. The rest of the paper is arranged so that each section treats a certain topic corresponding to one or more steps in the proof of Theorem 2.1.

Apart from conditions (V) , (G) , (UE) , and (LE) described in Section 2, we introduce here some more lettered conditions that are widely used in the proof.

We say that the *Faber-Krahn inequality* holds on (Γ, μ) if, for some positive exponent ν ,

$$\lambda_1(A) \geq c\mu(A)^{-1/\nu} \quad (FK)$$

for all nonempty finite sets $A \subset \Gamma$. In particular, (FK) holds in $\mathbb{Z}^D$ with $\nu = D/2$. If Γ is infinite and connected and if μ is the standard weight on Γ , then (FK) automatically holds with $\nu = 1/2$ (see [9, Prop. 2.5]). We are interested in (FK) with $\nu = \alpha/\beta$, where α and β are the parameters from (UE) and (LE) , in which case we have $\nu > 1$.

An easy consequence of (UE) is the *diagonal upper estimate*

$$p_n(x, x) \leq Cn^{-\alpha/\beta} \quad (DUE)$$

for all $x \in \Gamma$ and $n \geq 1$.

Consider the following estimates for the mean exit time and the exit probability:

$$E(x, R) \simeq R^\beta \quad (E)$$

for all $x \in \Gamma$, $R \geq 1$, and

$$\Psi_n(x, R) \leq C \exp \left(- \left(\frac{R^\beta}{Cn} \right)^{1/(\beta-1)} \right) \quad (\Psi)$$

for all $x \in \Gamma$, $R > 0$, and $n \geq 1$. For example, (E) and (Ψ) hold in $\mathbb{Z}^D$ with $\beta = 2$.

Part $(V) + (G) \implies (UE)$ of Theorem 2.1 is proved by the following chain of implications:

$$\begin{array}{ccc}
 \boxed{(V) \quad + \quad (G)} & & \\
 \Downarrow_{\text{Prop. 5.5}} & & \Downarrow_{\text{Prop. 6.3}} \\
 (FK) & & (E) \\
 \Downarrow_{\text{Prop. 5.1}} & & \Downarrow_{\text{Prop. 7.1}} \\
 (DUE) & & (\Psi) \\
 \underbrace{\hspace{10em}} & & \\
 \Downarrow_{\text{Prop. 8.1}} & & \\
 \boxed{(UE)} & &
 \end{array}$$

The relations among exponents α , β , γ , and ν involved in all conditions are as follows:

$$\alpha - \beta = \gamma \quad \text{and} \quad \alpha/\beta = \nu.$$

Given (DUE) and (Ψ) , one easily obtains full upper bound (UE) using the approach of Barlow and Bass [6] (see Section 8). The method of obtaining Faber-Krahn inequality (FK) from (V) and (G) is based on ideas of G. Carron [14]. The implication $(FK) \implies (DUE)$ is a discrete modification of the approach of A. Grigor'yan [32]. The implication $(V) + (G) \implies (E)$ was originally proved by Telcs [59], and here we give a simpler proof for that.

The crucial part of the proof of upper estimate (UE) is the implication $(E) \implies (\Psi)$. The following nearly Gaussian estimate is true *always*, without assuming (E) or anything else:

$$\Psi_n(x, R) \leq C \frac{V(x, R)}{\mu(x)} \exp\left(-\frac{R^2}{Cn}\right) \quad (4.1)$$

(see [58] and [33, p. 355]). However, (4.1) is not good enough for us even if we neglect the factor $V(x, R)$ in front of the exponential. Indeed, the range of n , for which we apply (Ψ) , is $n > R$ (see the proof of Proposition 8.1). Assuming $\beta > 2$, we have in this range

$$\left(\frac{R^\beta}{n}\right)^{1/(\beta-1)} > \frac{R^2}{n},$$

so that (Ψ) is stronger than (4.1).

We provide here an entirely new argument for $(E) \implies (\Psi)$, which is based on an investigation of solutions of the equation $\Delta v = \lambda v$. Function v can be estimated by comparing it to $\Delta u = -1$ (and the latter is related to the mean exit time). On the other hand, function $(1 + \lambda)^n v(x)$ satisfies the discrete heat equation and hence can be compared to $\Psi_n^A(x)$ by using the parabolic comparison principle (see Section 7 for details). Another proof of $(E) \implies (\Psi)$ can be obtained by using the probabilistic method of Barlow and Bass [5], [6], [7].

Before we consider the proof of lower bound (LE) , let us introduce the following conditions.

The *near-diagonal lower estimate* is

$$p_n(x, y) + p_{n+1}(x, y) \geq cn^{-\alpha/\beta} \quad \text{if } d(x, y) \leq \delta n^{1/\beta} \quad (NLE)$$

for some positive constant δ . Obviously, (NLE) is equivalent to (LE) in the range $d(x, y) \leq \delta n^{1/\beta}$.

As an intermediate step, we use the following *diagonal lower estimate* for the killed random walk:

$$p_{2n}^{B(x, R)}(x, x) \geq cn^{-\alpha/\beta} \quad \text{if } n \leq \varepsilon R^\beta \quad (DLE)$$

for some positive constant ε .

We say that the *Harnack inequality* holds on (Γ, μ) if, for any ball $B(x, 2R) \subset \Gamma$ and for any nonnegative function u in $\overline{B(x, 2R)}$ which is harmonic in $B(x, 2R)$,

$$\max_{B(x, R)} u \leq H \min_{B(x, R)} u \quad (H)$$

for some constant $H \geq 1$. The Harnack inequality reflects a certain homogeneity of the graph. For example, it holds for $\mathbb{Z}^D$ with the standard weight but fails on the connected sum of two copies of $\mathbb{Z}^D$ as well as on a binary tree.

The scheme of the proof of $(V) + (G) \implies (LE)$ is shown on the diagram below. From the previous diagram, we already know that conditions (FK) and (E) follow from $(V) + (G)$, as well as the implications $(FK) \implies (DUE)$ and $(E) \implies (\Psi)$:

The central point in the diagram is Proposition 13.1, where (NLE) is obtained from (DUE) , (DLE) , (E) , and (H) . The proof goes through the intermediate steps that

are denoted here by [osc] and [deriv]. The former refers to oscillation inequality (11.3) obtained from (H) in Propositions 11.1 and 11.2, and the latter refers to upper estimate (12.5) for $|p_{n+2} - p_n|$ obtained from (DUE) in Proposition 12.3.

The idea of obtaining (NLE) by means of an elliptic Harnack inequality seems to have appeared independently in papers by P. Auscher [2], [3], Barlow and Bass [6], [7], [8], and W. Hebisch and L. Saloff-Coste [38]. Basically, one views the heat equation for the heat kernel as an elliptic equation

$$\Delta(p_n + p_{n+1}) = f, \quad \text{where } f = p_{n+2} - p_n.$$

The elliptic Harnack inequality and the upper bound for $E(x, r)$ allow one to estimate the oscillation of $p_n + p_{n+1}$ via f . (In the continuous setting, the latter argument is classical and is due to J. Moser [49].)

On the other hand, the on-diagonal upper bound for p_n implies a suitable estimate for the discrete time derivative $p_{n+2} - p_n$. The fact that (DUE) implies a certain estimate of the time derivative of the heat kernel is well known. In the context of manifolds it goes back to S. Cheng, Li, and Yau [17] and E. Davies [26], [27] (see also [34]); in the discrete setting it follows from the results of E. Carlen, S. Kusuoka, and D. Stroock [13] and T. Coulhon and Saloff-Coste [23]; and in the setting of fractals it is proved by Barlow and Bass [7].

Having an upper bound for the oscillation of $p_n + p_{n+1}$ and the on-diagonal lower bound for $p_n + p_{n+1}$, one obtains (NLE). The final step in the proof—the implication (NLE) + (V) $\implies$ (LE)—is done by using the classical chaining argument of Moser [50] and D. Aronson [1].

The method of obtaining (DLE) from (Ψ) and (V) used in Proposition 9.1 is well known. Its various modifications can be found in [6], [11], [21], [24], [48], [56], and possibly in other places.

The claim that Green kernel estimate (G) implies elliptic Harnack inequality (H) would not surprise experts. In the context of the uniformly elliptic operators in $\mathbb{R}^D$, this was first observed by E. Landis [46, p. 145–146] and then was elaborated by N. Krylov and M. Safonov [43] and E. Fabes and Stroock [29]. However, this claim becomes rather nontrivial for arbitrary graphs (and manifolds) because of topological difficulties. We provide here a new, simple, and general proof of the implication (G) $\implies$ (H), which is based on the potential theoretic approach of A. Boukricha [12].

Finally, the converse implication (UE) + (LE) $\implies$ (V) + (G) is quite straightforward and is proved in Proposition 15.1.

As a consequence of the above diagrams, we see that the following equivalence takes place:

$$(FK) + (V) + (E) + (H) \iff (UE) + (LE).$$

It is possible to show that this equivalence is also true for recurrent graphs. Furthermore, Faber-Krahn inequality (FK) turns out to follow from $(V) + (E) + (H)$, so that

$$(V) + (E) + (H) \iff (UE) + (LE). \quad (4.2)$$

Condition (H) ensures here a necessary homogeneity of the graph, whereas (V) and (E) provide the exponents α and β , respectively.

Another consequence of the proof is that

$$(V) + (UE) + (H) \iff (UE) + (LE) \quad (4.3)$$

(see Remark 15.1). There are a number of conditions given in terms of capacities, eigenvalues, and so on, which can replace (E) or (UE) in (4.2) and (4.3), respectively. In the presence of (V) and (H) , the purpose of the other condition is to recover the exponent β in (UE) and (LE) . Note that if $\beta = 2$, then (UE) in (4.3) can be replaced by (DUE) (cf. [38]).

The complete proofs of (4.2), (4.3), and other related statements will be given elsewhere.

5. The Faber-Krahn inequality and on-diagonal upper bounds

Recall that a *Faber-Krahn inequality* holds on (Γ, μ) if there are constants $c > 0$ and $\nu > 0$ such that, for all nonempty finite sets $A \subset \Gamma$,

$$\lambda_1(A) \geq c\mu(A)^{-1/\nu}. \quad (FK)$$

We discuss here relationships between eigenvalue estimates like (FK) and estimates of the Green kernel, heat kernel, and volume growth. The outcome is the following implications:

$$(V) + (G) \implies (FK) \implies (DUE),$$

which are contained in Propositions 5.5 and 5.1, respectively, and which constitute a part of the proof of Theorem 2.1.

PROPOSITION 5.1

Let (Γ, μ) satisfy (p_0) , and let ν be a positive number. Then the following conditions are equivalent:

- (a) *Faber-Krahn inequality (FK) ;*
- (b) *the on-diagonal heat kernel upper bound, for all $x \in \Gamma$ and $n \geq 1$,*

$$p_n(x, x) \leq Cn^{-\nu}; \quad (DUE)$$

(c) *the estimate of the level sets of the Green kernel, for all $x \in \Gamma$ and $t > 0$,*

$$\mu\{y : g(x, y) > t\} \leq Ct^{-\nu/(\nu-1)}, \quad (5.1)$$

provided $\nu > 1$.

The analogue of Proposition 5.1 for manifolds was proved by Carron [14]. The equivalence (a) $\iff$ (b) was also proved in [32] for heat kernels on manifolds, and in [20, Prop. V.1] for random walks satisfying in addition the condition $\inf_x P(x, x) > 0$.

We provide detailed proof only for the implications (a) $\implies$ (b) and (c) $\implies$ (a) which we use in this paper. The implication (b) $\implies$ (c) can be proved in the following way. By a theorem of N. Varopoulos [63], (DUE) implies a Sobolev inequality. Then one applies an argument of [14, Prop. 1.14] (adapted to the discrete setting) to show that (5.1) follows from the Sobolev inequality.

Note that our proof of (a) $\implies$ (b) goes through for any $\nu > 0$. If $\nu > 1$, then one could apply the approach of [14] using a Sobolev inequality as an intermediate step between (a) and (b). In general, we use instead a Nash-type inequality that is obtained in the following lemma.

LEMMA 5.2

Let (Γ, μ) be a weighted graph (which is not necessarily connected). Assume that, for any nonempty finite set $A \subset \Gamma$,

$$\lambda_1(A) \geq \Lambda(\mu(A)), \quad (5.2)$$

where $\Lambda(\cdot)$ is a nonnegative nonincreasing function on $(0, \infty)$. Let $f(x)$ be a non-negative function on Γ with finite support. Denote

$$\sum_{x \in \Gamma} f(x)\mu(x) = a \quad \text{and} \quad \sum_{x \in \Gamma} f^2(x)\mu(x) = b.$$

Then, for any $s > 0$,

$$\frac{1}{2} \sum_{x \sim y} (\nabla_{xy} f)^2 \mu_{xy} \geq (b - 2sa) \Lambda(a/s). \quad (5.3)$$

Proof

If $b - 2sa < 0$, then (5.3) trivially holds. So, we can assume in the sequel that

$$s \leq \frac{b}{2a}. \quad (5.4)$$

Since $b \leq a \max f$, (5.4) implies $s < \max f$ and, therefore, the set

$$A_s = \{x \in \Gamma : f(x) > s\}$$

Figure 2. Sets A and A_s

is nonempty (see Figure 2).

Consider function $h = (f - s)_+$. This function belongs to $c_0(A_s)$ whence we obtain, by variational property (3.3) of eigenvalues,

$$\frac{1}{2} \sum_{x \sim y} (\nabla_{xy} h)^2 \mu_{xy} \geq \lambda_1(A_s) \sum_{x \in \Gamma} h^2(x) \mu(x). \quad (5.5)$$

Let us estimate all terms in (5.5) via f . We start with the obvious inequality

$$f^2 \leq (f - s)_+^2 + 2sf = h^2 + 2sf,$$

which holds for any $s \geq 0$. It implies $h^2 \geq f^2 - 2sf$ whence

$$\sum_{x \in \Gamma} h^2(x) \mu(x) \geq b - 2sa. \quad (5.6)$$

The definition of A_s implies $\mu(A_s) \leq a/s$ whence, by (5.2),

$$\lambda_1(A_s) \geq \Lambda(\mu(A_s)) \geq \Lambda(a/s). \quad (5.7)$$

Clearly, we also have

$$\sum_{x \sim y} (\nabla_{xy} h)^2 \mu_{xy} \leq \sum_{x \sim y} (\nabla_{xy} f)^2 \mu_{xy}.$$

Combining this with (5.7), (5.6), and (5.5), we obtain (5.3). $\square$

We apply Lemma 5.2 for function $\Lambda(v) = cv^{-1/v}$. Choosing $s = b/(4a)$ in (5.3), we obtain

$$\frac{1}{2} \sum_{x \sim y} (\nabla_{xy} f)^2 \mu_{xy} \geq c a^{-2/v} b^{1+1/v}. \quad (5.8)$$

This is a discrete version of the *Nash inequality* (cf. [51], [13]).

Proof of (a) $\implies$ (b) in Proposition 5.1

Step 1. Let f be a nonnegative function on Γ with finite support. Denote, for simplicity,

$$b = \sum_{x \in \Gamma} f^2(x) \mu(x) \quad \text{and} \quad b' = \sum_{x \in \Gamma} [Pf(x)]^2 \mu(x),$$

where P is the Markov operator of (Γ, μ) . Then we have

$$b - b' = (f, f)_{L^2(\Gamma, \mu)} - (Pf, Pf)_{L^2(\Gamma, \mu)} = (f, (I - P_2)f)_{L^2(\Gamma, \mu)}.$$

Clearly, $Q := P_2$ is also a Markov operator on Γ reversible with respect to μ , and it is associated with another structure of a weighted graph on the set Γ . Denote this weighted graph by (Γ^*, μ^*) . As a set, Γ^* coincides with Γ , and the measures μ and μ^* on vertices are the same. On the other hand, points x, y are connected by an edge on Γ^* if there is a path of length 2 from x to y in Γ , and the weight μ_{xy}^* on edges of Γ^* is defined by

$$\mu_{xy}^* = Q(x, y) \mu(x).$$

Denote by Δ^* the Laplace operator of (Γ^*, μ^*) . Then $\Delta^* = P_2 - I$ and, by Green formula (3.2),

$$b - b' = - \sum_{x \in \Gamma} f(x) \Delta^* f(x) \mu(x) = \frac{1}{2} \sum_{x, y \in \Gamma} (\nabla_{xy} f)^2 \mu_{xy}^*. \quad (5.9)$$

Step 2. If A is a nonempty finite subset of Γ , then [22, Lem. 4.3] says that*

$$\lambda_1^*(A) \geq \lambda_1(\bar{A}), \quad (5.10)$$

where $\lambda_1^*(A)$ is the first eigenvalue of $-\Delta_A^*$. By Faber-Krahn inequality (FK) for the graph (Γ, μ) , we obtain

$$\lambda_1^*(A) \geq c \mu(\bar{A})^{-1/\nu}. \quad (5.11)$$

Since (p_0) and Proposition 3.1 imply

$$\mu(\bar{A}) \leq \sum_{x \in A} V(x, 2) \leq C \sum_{x \in A} \mu(x) = C \mu(A) = C \mu^*(A),$$

(5.11) yields (FK) for the graph (Γ^*, μ^*) .

Remark 5.1

The only place where (p_0) is used in the proof of $(a) \implies (b)$ is to ensure that $\mu(\bar{A}) \leq C \mu(A)$. If this inequality holds for another reason, then the rest of the proof goes in the same way.

* The proof of (5.10) is based on variational property (3.3) and on the fact that all eigenvalues of $-\Delta_A$ belong to the interval $[\lambda_1(A), 2 - \lambda_1(A)]$.

Step 3. For some fixed $y \in \Gamma$, denote $f_n(x) = p_n(x, y)$ and

$$b_n = \sum_{x \in \Gamma} f_n^2(x) \mu(x) = p_{2n}(y, y).$$

Then $f_{n+1} = Pf_n$ and we obtain, by (5.9),

$$b_n - b_{n+1} = \frac{1}{2} \sum_{x, y \in \Gamma} (\nabla_{xy} f_n)^2 \mu_{xy}^*.$$

The graph (Γ^*, μ^*) satisfies (FK) so that Lemma 5.2 can be applied. Since

$$\sum_{x \in \Gamma} f_n(x) \mu(x) = \sum_{x \in \Gamma} P_n(x, y) = 1,$$

(5.8) yields

$$\frac{1}{2} \sum_{x, y \in \Gamma} (\nabla_{xy} f_n)^2 \mu_{xy}^* \geq c b_n^{1+1/\nu},$$

whence

$$b_n - b_{n+1} \geq c b_n^{1+1/\nu}. \quad (5.12)$$

In particular, we see that $b_n > b_{n+1}$.

Next we apply an elementary inequality

$$v(x - y) \geq \frac{x^\nu - y^\nu}{x^{\nu-1} + y^{\nu-1}}, \quad (5.13)$$

which is true for all $x > y > 0$ and $\nu > 0$. Taking $x = b_{n+1}^{-1/\nu}$ and $y = b_n^{-1/\nu}$, we obtain, from (5.13) and (5.12),

$$v(b_{n+1}^{-1/\nu} - b_n^{-1/\nu}) \geq \frac{b_{n+1}^{-1} - b_n^{-1}}{b_{n+1}^{-(\nu-1)/\nu} + b_n^{-(\nu-1)/\nu}} = \frac{b_n - b_{n+1}}{b_{n+1}^{1/\nu} b_n + b_n^{1/\nu} b_{n+1}} \geq \frac{c b_n^{1+1/\nu}}{2 b_n^{1+1/\nu}} = \frac{c}{2},$$

whence

$$b_{n+1}^{-1/\nu} - b_n^{-1/\nu} \geq \frac{c}{2\nu} = \text{const.}$$

Summing up this inequality in n , we conclude that $b_n^{-1/\nu} \geq cn$ and $b_n \leq Cn^{-\nu}$.

Since $b_n = p_{2n}(y, y)$, we have proved that, for all $y \in \Gamma$ and $n \geq 1$,

$$p_{2n}(y, y) \leq Cn^{-\nu}, \quad (5.14)$$

which is (DUE) for all *even* times.

Step 4. By the semigroup identity, we have, for any $0 < k < m$,

$$p_m(x, y) = \sum_{z \in \Gamma} p_{m-k}(x, z) p_k(z, y) \mu(z). \quad (5.15)$$

In particular, if $m = 2n$, $k = n$, and $y = x$, then

$$p_{2n}(x, x) = \sum_{z \in \Gamma} p_n^2(x, z) \mu(z). \quad (5.16)$$

On the other hand, (5.15), the Cauchy-Schwarz inequality, and (5.16) imply

$$\begin{aligned} p_{2n}(x, y) &= \sum_{z \in \Gamma} p_n(x, z) p_n(z, y) \mu(z) \\ &\leq \left[\sum_{z \in \Gamma} p_n^2(x, z) \mu(x) \right]^{1/2} \left[\sum_{z \in \Gamma} p_n^2(y, z) \mu(z) \right]^{1/2}, \end{aligned}$$

whence

$$p_{2n}(x, y) \leq p_{2n}(x, x)^{1/2} p_{2n}(y, y)^{1/2}. \quad (5.17)$$

Together with (5.14), this yields $p_{2n}(x, y) \leq C n^{-\nu}$ for all $x, y \in \Gamma$. This implies (DUE) also for odd times if we observe that, by (5.15) and (2.2),

$$p_{2n+1}(x, y) = \sum_{z \in \Gamma} p_{2n}(x, z) P(z, y) \leq \max_{z \in \Gamma} p_{2n}(x, z). \quad (5.18)$$

□

Proof of (c) $\Rightarrow$ (a) in Proposition 5.1

Let A be a nonempty finite subset of Γ , and let $f \in c_0(A)$ be the first eigenfunction of $-\Delta_A$. We may assume that $f \geq 0$. Let us normalize f so that $\max f = 1$, and let $x_0 \in A$ be the maximum point of f . The equation $-\Delta_A f = \lambda_1(A) f$ implies, by (3.7),

$$f(x) = \lambda_1(A) \sum_{y \in A} G_A(x, y) f(y),$$

whence, for $x = x_0$,

$$1 = \lambda_1(A) \sum_{y \in A} G_A(x_0, y) f(y) \leq \lambda_1(A) \sum_{y \in A} G_A(x_0, y)$$

and

$$\lambda_1(A) \geq \left(\max_{x \in A} \sum_{y \in A} G_A(x, y) \right)^{-1}. \quad (5.19)$$

On the other hand, for any $x \in A$,

$$\sum_{y \in A} G_A(x, y) = \sum_{y \in A} g_A(x, y) \mu(y) = \int_0^\infty \mu \{g_A(x, \cdot) > t\} dt.$$

Fix some $t_0 > 0$, and estimate the integral above using (5.1), $g_A \leq g$, and the fact that

$$\mu \{g_A(x, \cdot) > t\} \leq \mu(A).$$

Then we obtain

$$\sum_{y \in A} G_A(x, y) \leq \int_0^{t_0} \mu(A) dt + \int_{t_0}^\infty C t^{-v/(v-1)} dt = \mu(A)t_0 + C t_0^{-1/(v-1)}.$$

Let us choose $t_0 \simeq \mu(A)^{-(v-1)/v}$ to equate the two terms on the right-hand side, whence

$$\sum_{y \in A} G_A(x, y) \leq C \mu(A)^{1/v}. \quad (5.20)$$

Finally, (5.20) and (5.19) imply (FK). □

The second result of this section is preceded by two lemmas. We say that a weighted graph (Γ, μ) satisfies the *doubling volume condition* if

$$V(x, 2R) \leq C V(x, R), \quad \forall x \in \Gamma, R > 0. \quad (D)$$

Clearly, (D) is a weaker assumption than (V).

LEMMA 5.3

If (Γ, μ) satisfies (D), then, for all $x \in \Gamma$ and $R > 0$,

$$\lambda(x, R) \leq \frac{C}{R^2}. \quad (5.21)$$

Proof

Let us apply variational property (3.3) with the test function

$$f(y) = (R - d(x, y))_+ \in c_0(B(x, R)).$$

Since $|\nabla_{yz} f| \leq 1$, (3.3) and (D) imply

$$\lambda(x, R) \leq \frac{(1/2) \sum_{y \sim z} (\nabla_{yz} f)^2 \mu_{yz}}{\sum_y f^2(y) \mu(y)} \leq \frac{C V(x, R)}{R^2 V(x, R/2)} \leq \frac{C'}{R^2},$$

which was to be proved. □

The next lemma was proved in [59], but we give here a shorter proof.

LEMMA 5.4

Let (Γ, μ) satisfy (p_0) . If (V) and (G) hold, with some positive parameters α and γ , then $\alpha - \gamma \geq 2$.

Proof

By (5.19), we have

$$\lambda(x, R)^{-1} \leq \max_{y \in B(x, R)} \sum_{z \in B(x, 2R)} G(y, z). \quad (5.22)$$

By (G) and Proposition 3.2, $G(y, y)$ is uniformly bounded from above. Using (G) to estimate $G(y, z)$ for $y \neq z$ and (V) , we obtain

$$\begin{aligned} \sum_{z \in B(y, 2R)} G(y, z) &= G(y, y) + \sum_{i=-1}^{\lceil \log_2 R \rceil} \sum_{z \in B(y, 2^{-i}R) \setminus B(y, 2^{-i-1}R)} g(y, z) \mu(z) \\ &\leq C + C \sum_{i=-1}^{\lceil \log_2 R \rceil} (2^{-i}R)^{-\gamma} V(y, 2^{-i}R) \\ &\leq C \left[1 + \sum_{i=-1}^{\lceil \log_2 R \rceil} (2^{-i}R)^{\alpha-\gamma} \right]. \end{aligned} \quad (5.23)$$

A straightforward computation of sum (5.23) yields, for large R ,

$$\sum_{z \in B(y, 2R)} G(y, z) \leq C \begin{cases} R^{\alpha-\gamma}, & \alpha > \gamma, \\ \log_2 R, & \alpha = \gamma, \\ 1, & \alpha < \gamma. \end{cases} \quad (5.24)$$

Combining (5.22) and (5.24), we obtain

$$\lambda(x, R) \geq c \begin{cases} R^{-(\alpha-\gamma)}, & \alpha > \gamma, \\ (\log_2 R)^{-1}, & \alpha = \gamma, \\ 1, & \alpha < \gamma. \end{cases} \quad (5.25)$$

By Lemma 5.3, we have (5.21), which together with (5.25) implies $\alpha - \gamma \geq 2$. $\square$

PROPOSITION 5.5

Let (Γ, μ) satisfy (p_0) . If (V) and (G) hold with some positive parameters α and γ , then Faber-Krahn inequality (FK) holds with the parameter $\nu = \alpha/(\alpha - \gamma)$.

Proof

Note that, by Lemma 5.4, we have $\alpha > \gamma$ so that ν is positive and, moreover, $\nu > 1$. Let us verify that

$$\mu\{y : g(x, y) > t\} \leq \text{const } t^{-\alpha/\gamma}. \quad (5.26)$$

Then (5.1) would follow with $\nu = \alpha/(\alpha - \gamma)$, which implies (FK) , by Proposition 5.1.

The upper bound in (G) and (p_0) implies that, for all x, y (including the case $x = y$; see Proposition 3.2),

$$g(x, y) \leq C \min(1, d(x, y)^{-\gamma}). \quad (5.27)$$

If $t \geq C$, then the set $\{y : g(x, y) > t\}$ is empty, and (5.26) is trivially true.

Assume now that $t \leq C$. Then (5.27) implies

$$\mu\{y : g(x, y) > t\} \leq \mu\{y : d(x, y) < (t/C)^{-1/\gamma}\} = V(x, (t/C)^{-1/\gamma}).$$

Since $R := (t/C)^{-1/\gamma} \geq 1$, we can apply here the upper bound from (V) and obtain (5.26). $\square$

6. The mean exit time and the Green kernel

The purpose of this section is to verify part $(V) + (G) \implies (E)$ of the proof of Theorem 2.1. Recall that (E) stands for the condition

$$E(x, R) \simeq R^\beta, \quad \forall x \in \Gamma, \quad R \geq 1. \quad (E)$$

Alongside the mean exit time $E_A(x)$, consider the *maximal* mean exit time $\overline{E}_A$ defined by

$$\overline{E}_A := \sup_y E_A(y). \quad (6.1)$$

If $A = B(x, R)$, then we write $\overline{E}(x, R) := \overline{E}_{B(x, R)}$. We also use the following hypothesis:

$$\overline{E}(x, R) \leq CE(x, R), \quad \forall x \in \Gamma, \quad R > 0. \quad (\overline{E})$$

PROPOSITION 6.1

The upper bound in (E) implies, for all $x \in \Gamma$ and $R \geq 1$,

$$\overline{E}(x, R) \leq CR^\beta. \quad (6.2)$$

The lower bound in (E) implies

$$\overline{E}(x, R) \geq cR^\beta. \quad (6.3)$$

Consequently, (E) implies $(\overline{E})$ and

$$\overline{E}(x, R) \simeq R^\beta. \quad (6.4)$$

Proof

To show (6.2), let us observe that, for any point $y \in B(x, R)$, we have $B(x, R) \subset B(y, 2R)$, whence

$$\begin{aligned}\bar{E}(x, R) &= \sup_{y \in B(x, R)} E_{B(x, R)}(y) \leq \sup_{y \in B(x, R)} E_{B(y, 2R)}(y) \\ &= \sup_{y \in B(x, R)} E(y, 2R) \leq CR^\beta.\end{aligned}$$

Lower bound (6.3) is obvious by $E \leq \bar{E}$. Finally, $(\bar{E})$ follows from (E) and (6.4) if $R \geq 1$, and $(\bar{E})$ holds trivially if $R < 1$. $\square$

PROPOSITION 6.2

For any nonempty finite set $A \subset \Gamma$, we have

$$\lambda_1(A) \geq (\bar{E}_A)^{-1}. \quad (6.5)$$

Proof

Indeed, this is a combination of (5.19) and the definition of $\bar{E}$ (see (3.8) and (6.1)). $\square$

The next statement was proved in [59].

PROPOSITION 6.3

Let (Γ, μ) satisfy (p_0) . If (V) and (G) hold, with some positive parameters α and γ , then (E) holds as well with $\beta = \alpha - \gamma$.

Proof

Denote $A = B(x, R)$. Applying (3.8), the obvious inequality $g_A \leq g$, as well as (V) and (G), we obtain (cf. (5.23) and (5.24))

$$E(x, R) = \sum_{y \in A} g_A(x, y) \mu(y) \leq \sum_{y \in A} g(x, y) \mu(y) \leq CR^{\alpha-\gamma}.$$

Observe that, by Lemma 5.4, we already know that $\alpha > \gamma$.

For the lower bound of $E(x, R)$, let us prove that

$$g_A(x, y) \geq c d(x, y)^{-\gamma}, \quad \forall y \in B(x, \varepsilon R) \setminus \{x\}, \quad (6.6)$$

provided $\varepsilon > 0$ is small enough. Consider the function

$$u(y) = g(x, y) - g_A(x, y)$$

which is harmonic in A . By the maximum principle, its maximum is attained at the boundary of A , whence, by (G),

$$0 \leq u(y) \leq CR^{-\gamma}.$$

Therefore,

$$g_A(x, y) = g(x, y) - u(y) \geq c d(x, y)^{-\gamma} - C R^{-\gamma}. \quad (6.7)$$

If R is large enough and if $d(x, y) \leq \varepsilon R$ with a small enough ε , then the second term in (6.7) is absorbed by the first one, whence (6.6) follows.

Summing up (6.6) over y , we obtain (cf. (5.23) and (5.24))

$$E(x, R) = \sum_{y \in A} g_A(x, y) \mu(y) \geq \sum_{y \in B(x, \varepsilon R) \setminus \{x\}} g_A(x, y) \mu(y) \geq c R^{\alpha - \gamma}.$$

If R is not big enough, then the above argument does not work. However, in this case we argue as follows. If the random walk starts at x , then $T_{B(x, R)} \geq R$. Hence, we always have $E(x, R) = \mathbb{E}_x(T_{B(x, R)}) \geq R$ which yields the lower bound in (E), provided $R \leq \text{const}$. $\square$

Assuming that (V) and (E) hold, there are the following general relations between the exponents α and β : if the graph is transient, then $2 \leq \beta \leq \alpha$; and if it is recurrent, then $2 \leq \beta \leq \alpha + 1$ (see [59]; see also [60], [61] for various definitions of dimensions of graphs).

7. Sub-Gaussian term

The following statement is crucial for obtaining the off-diagonal upper bound of the heat kernel. It contains the part (E) $\implies$ (Ψ) of the proof of Theorem 2.1.

PROPOSITION 7.1

Assume that the graph (Γ, μ) possesses property (E). Then, for all $x \in \Gamma$, $R > 0$, and $n \geq 1$, we have

$$\Psi_n(x, R) \leq C \exp \left(- \left(\frac{R^\beta}{Cn} \right)^{1/(\beta-1)} \right). \quad (\Psi)$$

We start with the following lemma.

LEMMA 7.2

Assume that hypothesis $(\overline{E})$ holds on (Γ, μ) . Let $A = B(x_0, r)$ be an arbitrary ball on Γ , and let v be a function on $\overline{A}$ such that $0 \leq v \leq 1$. Suppose that v satisfies in A the equation

$$\Delta v = \lambda v, \quad (7.1)$$

where λ is a constant such that

$$\lambda \geq (\overline{E}_A)^{-1}. \quad (7.2)$$

Figure 3. The value of the function v at the point x_0 does not exceed $1 - \varepsilon$

Then

$$v(x_0) \leq 1 - \varepsilon, \quad (7.3)$$

where $\varepsilon > 0$ depends on the constants in hypothesis $(\bar{E})$ (see Figure 3).

Proof

Denote for simplicity $u(x) = E_A(x)$, and recall that $u \in c_0(A)$ and $\Delta u = -1$ in A (cf. (3.9)). Also, denote

$$\lambda_0 := (\bar{E}_A)^{-1} = \frac{1}{\max u}.$$

Consider the function $w = 1 - (\lambda_0/2)u$. Then $1/2 \leq w \leq 1$ and, in A ,

$$\Delta w = \frac{\lambda_0}{2} \leq \lambda_0 w \leq \lambda w.$$

Since $v \leq 1$ and $w = 1$ outside A , the maximum principle for the operator $\Delta - \lambda$ implies that $v \leq w$ in A . In particular,

$$v(x_0) \leq w(x_0) = 1 - \frac{\lambda_0}{2}u(x_0) \leq 1 - \frac{u(x_0)}{2 \max u}.$$

Hypothesis $(\bar{E})$ yields

$$\frac{u(x_0)}{\max u} = \frac{E(x_0, r)}{\bar{E}(x_0, r)} \geq c,$$

whence (7.3) follows. $\square$

LEMMA 7.3

Assume that (Γ, μ) satisfies (E) . Let $A = B(x_0, R)$ be an arbitrary ball on Γ , and let v be a function on $\bar{A}$ such that $0 \leq v \leq 1$. If v satisfies, in A , equation (7.1) with a constant λ such that

$$CR^{-\beta} \leq \lambda < \bar{\lambda}, \quad (7.4)$$

Figure 4. The points x_i where $v(x)$ takes the maximum values

then

$$v(x_0) \leq \exp \left(-c\lambda^{1/\beta} R \right). \quad (7.5)$$

Here $\bar{\lambda}$ is an arbitrary constant, C is some constant depending on condition (E), and $c > 0$ is some constant depending on $\bar{\lambda}$ and on condition (E).

Proof

Condition (E) implies $(\bar{E})$ and $\bar{E}(x, R) \simeq R^\beta$ (see Proposition 6.1). Choose the constant C in (7.4) so big that the lower bound in (7.4) implies $\lambda \geq \bar{E}(x, R)^{-1}$. Then, by Lemma 7.2, we obtain $v(x_0) \leq 1 - \varepsilon$. If we have, in addition,

$$\lambda^{1/\beta} R \leq \text{const}, \quad (7.6)$$

then (7.5) is trivially satisfied. In particular, if R is in the bounded range, then (7.6) is true because λ is bounded from above by (7.4).

Hence, we may assume in the sequel that

$$R > C' \quad \text{and} \quad \lambda > C'' R^{-\beta}, \quad (7.7)$$

with large enough constants C' and C'' (in particular, $C'' \gg C$). The point of the present lemma is that it improves the previous one for this range of R and λ . Choose a number r from the equation $\lambda = Cr^{-\beta}$, where C is the same constant as in (7.4). The above argument shows that Lemma 7.2 applies in any ball of radius r . Let x_i , $i \geq 1$, be a point in the ball $B(x_0, (r+1)i)$ in which v takes the maximum value in this ball, and denote $m_i = v(x_i)$ (see Figure 4). For $i = 0$, we set $m_0 = v(x_0)$.

For each $i \geq 0$, consider the ball $A_i = B(x_i, r)$. Since

$$\bar{A}_i \subset B(x_i, r+1) \subset B(x_0, (r+1)(i+1)),$$

we have

$$\max_{\overline{A}_i} v \leq m_{i+1}.$$

Applying Lemma 7.2 to the function v/m_{i+1} in the ball A_i , we obtain

$$m_i \leq (1 - \varepsilon)m_{i+1}.$$

Iterating this inequality $k := \lfloor R/(r+1) \rfloor$ times and using $m_k \leq 1$, we conclude

$$v(x_0) = m_0 \leq (1 - \varepsilon)^k. \quad (7.8)$$

By conditions (7.7) and (7.4) and by the choice of r , we have

$$k \simeq \frac{R}{r} \simeq \lambda^{1/\beta} R,$$

so that (7.8) implies (7.5). □

LEMMA 7.4

Assume that (Γ, μ) satisfies (E). Let $A = B(x_0, R)$ be an arbitrary ball on Γ , and let $w_n(x)$ be a function in $\overline{A} \times \mathbb{N}$ such that $0 \leq w \leq 1$. Suppose that w solves in $A \times \mathbb{N}$ the heat equation

$$w_{n+1} - w_n = \Delta w_n \quad (7.9)$$

with initial data $w_0 \equiv 0$ in A (see Figure 5). Then, for all $n \geq 1$,

$$w_n(x_0) \leq \exp \left(-c \left(\frac{R^\beta}{n} \right)^{1/(\beta-1)} + 1 \right). \quad (7.10)$$

Proof

First, consider two trivial cases. If $R^\beta \leq Cn$, then (7.10) is true just by $w \leq 1$, provided c is small enough. Since $\Delta w(x)$ depends only on the immediate neighbors of x , one gets by induction that $w_k(x) = 0$ for all $x \in B(x_0, R - k)$. Therefore, if $R > n$, then $w_n(x_0) = 0$, and (7.10) is true again.

Hence, we may assume in the sequel that, for a large enough C ,

$$Cn^{1/\beta} < R \leq n. \quad (7.11)$$

Fix some $\lambda > 0$, and find a function $v(x)$ on $\overline{A}$ solving the boundary value problem

$$\begin{cases} \Delta v = \lambda v & \text{in } A, \\ v = 1 & \text{in } \overline{A} \setminus A. \end{cases}$$

Figure 5. The value of the function w at the point (x_0, n) is affected by the initial value $w = 0$ and by the boundary condition $w \leq 1$

The function $u_n(x) := (1 + \lambda)^n v(x)$ solves heat equation (7.9) and satisfies the following boundary conditions: $u_n(x) \geq 1$ for $x \in \bar{A} \setminus A$ and $u_0(x) \geq 0$ for $x \in A$. By the parabolic comparison principle, we have $w \leq u$. Assume for a moment that λ satisfies hypothesis (7.4) of Lemma 7.3. Then we estimate $v(x_0)$ by (7.5) and obtain

$$w_n(x_0) \leq (1 + \lambda)^n v(x_0) \leq \exp\left(\lambda n - c\lambda^{1/\beta} R\right).$$

Now, choose λ from the condition $c\lambda^{1/\beta} R = 2\lambda n$; that is,

$$\lambda = \left(\frac{cR}{2n}\right)^{\beta/(\beta-1)}. \quad (7.12)$$

As follows from (7.11), this particular λ satisfies (7.4). Therefore, the above application of Lemma 7.3 is justified, and we obtain

$$w_n(x_0) \leq \exp(-\lambda n) = \exp\left(-c'\left(\frac{R^\beta}{n}\right)^{1/(\beta-1)}\right),$$

finishing the proof. □

Proof of Proposition 7.1

Denote $A = B(x_0, R)$. By (3.11), the function $w_n(x) := \Psi_n^A(x)$ satisfies all the hypotheses of Lemma 7.4. Hence, (Ψ) follows from (7.10). □

8. Off-diagonal upper bound of the heat kernel

Here we prove the following implication:

$$(FK) + (E) \implies (UE), \quad (8.1)$$

which finishes the proof of the heat kernel upper bound in Theorem 2.1. Indeed, together with the implications

$$(V) + (G) \xrightarrow{\text{Prop. 5.5}} (FK)$$

and

$$(V) + (G) \xrightarrow{\text{Prop. 6.3}} (E),$$

(8.1) yields the part $(V) + (G) \implies (UE)$ of Theorem 2.1.

PROPOSITION 8.1

On any graph (Γ, μ) , we have

$$(DUE) + (\Psi) \implies (UE). \quad (8.2)$$

In particular, if (p_0) holds on (Γ, μ) , then

$$(FK) + (E) \implies (UE). \quad (8.3)$$

Proof

By Proposition 5.1, (p_0) and (FK) imply (DUE) . By Proposition 7.1, (E) implies (Ψ) . Hence, implication (8.3) is a consequence of (8.2).

To prove (8.2), let us fix some points $x, y \in \Gamma$ and denote $r = d(x, y)/2$. Since balls $B(x, r)$ and $B(y, r)$ do not intersect, the semigroup identity (5.15) and the symmetry of the heat kernel imply, for any triple of nonnegative integers k, m, n such that $k + m = n$,

$$\begin{aligned} p_n(x, y) &\leq \sum_{z \notin B(x, r)} p_m(x, z) p_k(z, y) \mu(z) + \sum_{z \notin B(y, r)} p_m(x, z) p_k(z, y) \mu(z) \\ &\leq \sup_z p_k(z, y) \sum_{z \notin B(x, r)} P_m(x, z) + \sup_z p_m(x, z) \sum_{z \notin B(y, r)} P_k(y, z) \\ &= \sup_z p_k(y, z) \mathbb{P}_x(X_m \notin B(x, r)) + \sup_z p_m(x, z) \mathbb{P}_y(X_k \notin B(x, r)). \end{aligned}$$

As follows from definition (3.10) of Ψ ,

$$\mathbb{P}_x(X_m \notin B(x, r)) \leq \Psi_m(x, r).$$

Hence, we obtain the following general inequality, which is true for all reversible random walks:

$$p_n(x, y) \leq \sup_z p_k(y, z) \Psi_m(x, r) + \sup_z p_m(x, z) \Psi_k(y, r). \quad (8.4)$$

As follows from (5.17), diagonal upper bound (*DUE*) implies, for all $x, y \in \Gamma$,

$$p_n(x, y) \leq Cn^{-\alpha/\beta}, \quad (8.5)$$

provided n is even. Using inequality (5.18), we see that (8.5) also holds for odd n . Assuming $n \geq 2$, choosing $k \simeq m \simeq n/2$, and applying (8.5) and (Ψ) to estimate the right-hand side of (8.4), we obtain (*UE*). If $n = 1$, then (*UE*) follows trivially from (8.5) and the fact that $p_n(x, y) = 0$ whenever $d(x, y) > n$. $\square$

9. On-diagonal lower bound

In this section we prove part (Ψ) + (*V*) $\implies$ (*DLE*) of Theorem 2.1.

PROPOSITION 9.1

Assume that hypothesis (Ψ) holds on (Γ, μ) . For arbitrary $x \in \Gamma$ and $R > 0$, denote $A = B(x, R)$. Then the following on-diagonal lower bound is true:

$$p_{2n}^A(x, x) \geq \frac{c}{V(x, Cn^{1/\beta})}, \quad (9.1)$$

provided $n \leq \varepsilon R^\beta$, where ε is a sufficiently small positive constant depending only on the constants from (Ψ).

If in addition (*V*) holds, then

$$p_{2n}^A(x, x) \geq cn^{-\alpha/\beta}, \quad \forall n \leq \varepsilon R^\beta. \quad (DLE)$$

Remark 9.1

Since $p_{2n} \geq p_{2n}^A$ for any $A = B(x, R)$, inequality (*DLE*) implies $p_{2n}(x, x) \geq cn^{-\alpha/\beta}$ for all positive integers n .

Proof

Let us fix some $r \in (0, R)$ and denote $B = B(x, r)$. Since $p^B \leq p^A$, it suffices to prove (9.1) for p^B instead of p^A , for some $r < R$. Semigroup identity (5.15) for p^B and the Cauchy-Schwarz inequality imply

$$p_{2n}^B(x, x) = \sum_{z \in B} p_n^B(x, z)^2 \mu(z) \geq \frac{1}{\mu(B)} \left(\sum_{z \in B} p_n^B(x, z) \mu(z) \right)^2. \quad (9.2)$$

Let us observe that

$$\sum_{z \in B} p_n^B(\cdot, z) \mu(z) + \Psi_n^B(\cdot) = 1. \quad (9.3)$$

Indeed, the first term in (9.3) is the probability that the random walk X_k stays in B up to the time $k = n$, whereas Ψ_n^B is the probability of the opposite event.

By hypothesis (Ψ) , we have

$$\Psi_n^B(x) = \Psi_n(x, r) \leq C \exp \left(- \left(\frac{r^\beta}{Cn} \right)^{1/(\beta-1)} \right). \quad (9.4)$$

Choosing $r = Cn^{1/\beta}$ for large enough C and assuming $n \leq \varepsilon R^\beta$ for sufficiently small $\varepsilon > 0$ (the latter ensures $r < R$), we obtain, from (9.4), $\Psi_n(x, r) \leq 1/2$, whence, by (9.3),

$$\sum_{z \in B} p_n^B(x, z) \mu(z) \geq \frac{1}{2}.$$

Therefore, (9.2) yields

$$p_{2n}^B(x, x) \geq \frac{1/4}{V(x, r)} = \frac{1/4}{V(x, Cn^{1/\beta})},$$

finishing the proof. $\square$

10. The Harnack inequality and the Green kernel

Recall that the weighted graph (Γ, μ) satisfies the *elliptic Harnack inequality* if, for all $x \in \Gamma$, $R > 0$, and for any nonnegative function u in $\overline{B(x, 2R)}$ which is harmonic in $B(x, 2R)$,

$$\max_{B(x, R)} u \leq H \min_{B(x, R)} u \quad (H)$$

with some constant $H > 1$. In this section we establish that (H) is implied by condition (G) . Recall that the latter refers to

$$g(x, y) \simeq d(x, y)^{-\gamma}, \quad \forall x \neq y. \quad (G)$$

Consider the following *annulus* Harnack inequality for the Green kernel: for all $x \in \Gamma$ and $R > 1$,

$$\max_{y \in A(x, R)} g(x, y) \leq C \min_{y \in A(x, R)} g(x, y), \quad (HG)$$

where $A(x, R) := B(x, R) \setminus B(x, R/2)$.

PROPOSITION 10.1

Assume that (p_0) hold and the graph (Γ, μ) is transient. Then

$$(G) \implies (HG) \implies (H).$$

Since the implication $(G) \implies (HG)$ is obvious, we need to prove only the second implication. The main part of the proof is contained in the following lemma.

Figure 6. The sets $B = U_0$, $A = U_2 \setminus U_1$, and $U = U_3$

LEMMA 10.2

Let $U_0 \subset U_1 \subset U_2 \subset U_3$ be a sequence of finite sets in Γ such that $\overline{U_i} \subset U_{i+1}$, $i = 0, 1, 2$. Denote $A = U_2 \setminus U_1$, $B = U_0$, and $U = U_3$. Then, for any function u that is nonnegative in $\overline{U_2}$ and harmonic in U_2 , we have

$$\max_B u \leq H \min_B u, \quad (10.1)$$

where

$$H := \max_{x \in B} \max_{y \in B} \max_{z \in A} \frac{G_U(y, z)}{G_U(x, z)} \quad (10.2)$$

(see Figure 6).

Remark 10.1

Note that no a priori assumption has been made about the graph (Γ, μ) (except for connectedness and unboundedness). If the graph is transient, then, by exhausting Γ by a sequence of finite sets U , we can replace G_U in (10.2) by G . Note also that, without loss of generality, one can take $U_2 = \overline{U_1}$.

Proof

The following potential-theoretic argument is borrowed from [12]. We use the notation of Section 3. Given a nonnegative harmonic function u in U_2 , denote by S_u the following class of superharmonic functions:

$$S_u = \{v : v \geq 0 \text{ in } \overline{U}, \Delta v \leq 0 \text{ in } U, \text{ and } v \geq u \text{ in } \overline{U_1}\}.$$

Define the function w on $\overline{U}$ by

$$w(x) = \min \{v(x) : v \in S_u\}. \quad (10.3)$$

Figure 7. The function u , a function $v \in S_u$, and the function $w = \min_{S_u} v$. The latter is harmonic in U_1 and in $U \setminus \overline{U}_1$.

Clearly, $w \in S_u$. Since the function u itself is also in S_u , we have $w \leq u$ in $\overline{U}$. On the other hand, by definition of S_u , $w \geq u$ in $\overline{U}_1$, whence we see that $u = w$ in $\overline{U}_1$ (see Figure 7). In particular, it suffices to prove (10.1) for w instead of u .

Let us show that $w \in c_0(U)$. Indeed, let $v(x) = E_U(x)$. Then, by (3.9) and the strong minimum principle, v is superharmonic and strictly positive in U . Hence, for a large enough constant C , we have $Cv \geq u$ in $\overline{U}_1$, whence $Cv \in S_u$ and $w \leq Cv$. Since $v = 0$ in $\overline{U} \setminus U$, this implies $w = 0$ in $\overline{U} \setminus U$ and $w \in c_0(U)$.

Denote $f := -\Delta w$, and observe that $f \geq 0$ in U . Since $w \in c_0(U)$, we have, for any $x \in U$,

$$w(x) = \sum_{z \in U} G_U(x, z) f(z). \quad (10.4)$$

Next we prove that $f = 0$ outside A so that the summation in (10.4) can be restricted to $z \in A$. Given this, we obtain, for all $x, y \in B$,

$$\frac{w(y)}{w(x)} = \frac{\sum_{z \in A} G_U(y, z) f(z)}{\sum_{z \in A} G_U(x, z) f(z)} \leq H,$$

whence (10.1) follows.

We are left to verify that w is harmonic in U_1 and outside $\overline{U}_1$. Indeed, if $x \in U_1$, then

$$\Delta w(x) = \Delta u(x) = 0$$

because $w = u$ in $\overline{U}_1$. Let $\Delta w(x) \neq 0$ for some $x \in U \setminus \overline{U}_1$. Since w is superharmonic, we have $\Delta w(x) < 0$ and

$$w(x) > Pw(x) = \sum_{y \sim x} P(x, y) w(y).$$

Consider the function w' , which is equal to w everywhere in $\overline{U}$ except for the point x ; and w' at x is defined to satisfy

$$w'(x) = \sum_{y \sim x} P(x, y) w'(y).$$

Clearly, $w'(x) < w(x)$, and w' is superharmonic in U . Since $w' = w = u$ in $\overline{U}_1$, we have $w' \in S_u$. Hence, by definition (10.3) of w , $w \leq w'$ in $\overline{U}$, which contradicts $w(x) > w'(x)$. $\square$

Proof of Proposition 10.1

Now we assume (HG) and prove (H). Given any ball $B(x_0, 2R)$ of radius $R > 4$ and a nonnegative harmonic function u in $B(x_0, 2R)$, define the sequence of radii $R_0 = R$, $R_1 = 3R/2$, and $R_2 = 2R$, and denote $U_i = B(x_0, R_i)$ for $i = 0, 1, 2$ and $U_3 = \Gamma$. By Lemma 10.2, we have inequality (10.1), which implies (H), provided we can show that the Harnack constant H from (10.2) is bounded from above, uniformly in x_0 and R . Indeed, if $x, y \in B(x_0, R)$ and $z \in A = \overline{B(x_0, 2R)} \setminus B(x_0, 3R/2)$, then both distances $d(z, x)$ and $d(z, y)$ are between $R/2$ and $7R/2$. By iterating (HG) in the annuli centered at z , we obtain

$$\frac{G(y, z)}{G(x, z)} = \frac{g(z, y)}{g(z, x)} \leq \text{const},$$

whence we see that H is indeed uniformly bounded from above.

The condition $R > 4$, which we have imposed above, ensures that $\overline{U_i} \subset U_{i+1}$, which is required for Lemma 10.2. If $R \leq 4$, then (H) simply follows from (p_0) and Proposition 3.2. $\square$

11. Oscillation inequalities

For any nonempty finite set U and a function u on U , denote

$$\text{osc}_U u := \max_U u - \min_U u.$$

The purpose of this section is to prove estimate (11.3), which provides the step (H) $\implies$ [osc] of the proof of Theorem 2.1.

PROPOSITION 11.1

Assume that elliptic Harnack inequality (H) holds on (Γ, μ) . Then, for any $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon, H) < 1$ such that, for any ball $B(x, R)$ and for any function u defined in $\overline{B(x, R)}$ and harmonic in $B(x, R)$, we have

$$\text{osc}_{B(x, \sigma R)} u \leq \varepsilon \text{osc}_{B(x, R)} u. \quad (11.1)$$

Proof

Fix a ball $B(x, R)$, and denote for simplicity $B_r = B(x, r)$. Let us prove that, for any $r \in (0, R/3]$,

$$\operatorname{osc}_{B_r} u \leq (1 - \delta) \operatorname{osc}_{B_{3r}} u, \quad (11.2)$$

where $\delta = \delta(H) \in (0, 1)$. Then (11.1) follows from (11.2) by iterating.

If $r \leq 1$, then the left-hand side of (11.2) vanishes and (11.2) is trivially satisfied. If $r > 1$, then $\overline{B_{2r}} \subset B_{3r}$, and the function $u - \min_{B_{3r}} u$ is nonnegative in $\overline{B_{2r}}$ and harmonic in B_{2r} . Applying Harnack inequality (H) to this function, we obtain

$$\max_{B_r} u - \min_{B_{3r}} u \leq H \left(\min_{B_r} u - \min_{B_{3r}} u \right)$$

and

$$\operatorname{osc}_{B_r} u \leq (H - 1) \left(\min_{B_r} u - \min_{B_{3r}} u \right).$$

Similarly, we have

$$\operatorname{osc}_{B_r} u \leq (H - 1) \left(\max_{B_{3r}} u - \max_{B_r} u \right).$$

Summing up these two inequalities, we conclude

$$\operatorname{osc}_{B_r} u \leq C \left(\operatorname{osc}_{B_{3r}} u - \operatorname{osc}_{B_r} u \right),$$

whence (11.2) follows. $\square$

PROPOSITION 11.2

Assume that elliptic Harnack inequality (H) holds on (Γ, μ) . Let $u \in c_0(B(x, R))$ satisfy in $B(x, R)$ the equation $\Delta u = f$. Then, for any positive $r < R$,

$$\operatorname{osc}_{B(x, \sigma r)} u \leq 2 \left(\overline{E}(x, r) + \varepsilon \overline{E}(x, R) \right) \max |f|, \quad (11.3)$$

where σ and ε are the same as in Proposition 11.1.

Proof

Denote for simplicity $B_r = B(x, r)$. By definition of the Green function, we have

$$u(y) = - \sum_{z \in B_R} G_{B_R}(y, z) f(z),$$

whence, using (3.8), we obtain

$$\max |u| \leq \overline{E}(x, R) \max |f|.$$

Figure 8. The functions u and v in the case $f \leq 0$

Let $v \in c_0(B_r)$ solve the Dirichlet problem $\Delta v = f$ in B_r (see Figure 8). In the same way, we have

$$\max |v| \leq \overline{E}(x, r) \max |f|.$$

The function $w = u - v$ is harmonic in B_r whence, by Proposition 11.1,

$$\operatorname{osc}_{B_{\sigma r}} w \leq \varepsilon \operatorname{osc}_{B_r} w.$$

Since $w = u$ on $\overline{B_r} \setminus B_r$, the maximum principle implies that

$$\operatorname{osc}_{B_r} w = \operatorname{osc}_{\overline{B_r} \setminus B_r} w = \operatorname{osc}_{\overline{B_r} \setminus B_r} u \leq 2 \max |u|.$$

Hence,

$$\operatorname{osc}_{B_{\sigma r}} u \leq \operatorname{osc}_{B_{\sigma r}} v + \operatorname{osc}_{B_{\sigma r}} w \leq 2 \max |v| + 2\varepsilon \max |u| \leq 2 \left(\overline{E}(x, r) + \varepsilon \overline{E}(x, R) \right) \max |f|,$$

which was to be proved. $\square$

12. Time derivative of the heat kernel

Given a function $u_n(x)$ on $\Gamma \times \mathbb{N}$, by the “time derivative” of u we mean the difference

$$\partial_n u := u_{n+2} - u_n.$$

The main result of this section is Proposition 12.3, which provides upper bound (12.5) for $\partial_n p$ and thus constitutes the part (DUE) $\implies$ [deriv] of the proof of Theorem 2.1. The crucial point is that $\partial_n p$ decays as $n \rightarrow \infty$ faster than p_n .

The analogue of the time derivative in the discrete case is $\partial_n p = p_{n+2} - p_n$ rather than $p_{n+1} - p_n$. Indeed, in $\mathbb{Z}^D$ (as well as in any other bipartite graph), $p_n(x, x) = 0$ if n is odd. Therefore, the difference $p_{n+1}(x, x) - p_n(x, x)$ is equal either to $p_{n+1}(x, x)$ or to $-p_n(x, x)$, and hence it decays as $n \rightarrow \infty$ at the same rate as $p_n(x, x)$.

PROPOSITION 12.1

Let A be a nonempty finite subset of Γ , and let f be a function on A . Define

$$u_n(x) = P_n^A f(x).$$

Then, for all integers $1 \leq k \leq n$,

$$\|\partial_n u\|_{L^2(A, \mu)} \leq \frac{1}{k} \|u_{n-k}\|_{L^2(A, \mu)}.$$

Proof

The proof follows the argument from [17]. Let $\phi_1, \phi_2, \dots, \phi_{|A|}$ be the eigenfunctions of the Laplace operator $-\Delta_A$, and let $\lambda_1, \lambda_2, \dots, \lambda_{|A|}$ be the corresponding eigenvalues. Let us normalize ϕ_i 's to form an orthonormal basis in $L^2(A, \mu)$. The function f can be expanded in this basis:

$$f = \sum_i c_i \phi_i.$$

Since $P^A = I - (-\Delta_A)$, we obtain

$$u_n = \sum_i \rho_i^n \phi_i, \quad (12.1)$$

where $\rho_i := 1 - \lambda_i$ are eigenvalues of the Markov operator P^A .

From (12.1), we obtain

$$u_n - u_{n+2} = \sum_i (1 - \rho_i^2) \rho_i^n \phi_i$$

and

$$\|u_n - u_{n+2}\|_{L^2(A, \mu)}^2 = \sum_i (1 - \rho_i^2)^2 \rho_i^{2n}. \quad (12.2)$$

Note that $|\rho_i| \leq 1$ and hence $\rho_i^2 \in [0, 1]$. For any $a \in [0, 1]$, we have

$$1 \geq (1 + a + a^2 + \dots + a^k)(1 - a) \geq ka^k(1 - a),$$

whence

$$(1 - a)a^k \leq \frac{1}{k}.$$

Applying this inequality for $a = \rho_i^2$, we obtain, from (12.2),

$$\|u_n - u_{n+2}\|_{L^2(A, \mu)}^2 \leq \frac{1}{k^2} \sum_i \rho_i^{2(n-k)} = \frac{1}{k^2} \|u_{n-k}\|_{L^2(A, \mu)}^2,$$

which was to be proved. □

PROPOSITION 12.2

Let A be a nonempty finite subset of Γ . Then, for all $x, y \in A$,

$$\left| \partial_n p^A(x, y) \right| \leq \frac{1}{k} \sqrt{p_{2m}^A(x, x) p_{2(n-m-k)}^A(y, y)} \quad (12.3)$$

for all positive integers n, m, k such that $m + k \leq n$.

Proof

From semigroup identity (5.15) for p^A , we obtain

$$\partial_n p^A(x, y) = \sum_{z \in A} p_m^A(x, z) \partial_{n-m} p^A(z, y) \mu(z),$$

whence

$$\left| \partial_n p^A(x, y) \right| \leq \left\| p_m^A(x, \cdot) \right\|_{L^2(A, \mu)} \left\| \partial_{n-m} p^A(y, \cdot) \right\|_{L^2(A, \mu)}.$$

By Proposition 12.1,

$$\left\| \partial_{n-m} p^A(y, \cdot) \right\|_{L^2(A, \mu)} \leq \frac{1}{k} \left\| p_{n-m-k}^A(y, \cdot) \right\|_{L^2(A, \mu)}$$

for any $1 \leq k \leq n - m$. Since

$$\left\| p_m^A(x, \cdot) \right\|_{L^2(A, \mu)}^2 = \sum_{z \in A} p_m^A(x, z)^2 \mu(z) = p_{2m}^A(x, x),$$

we obtain (12.3). □

PROPOSITION 12.3

Suppose that (DUE) holds; that is, for all $x \in \Gamma$ and $n \geq 1$,

$$p_n(x, x) \leq Cn^{-\nu}. \quad (12.4)$$

Then, for all $x, y \in \Gamma$ and $n \geq 1$,

$$|\partial_n p(x, y)| \leq Cn^{-\nu-1}. \quad (12.5)$$

Proof

First, assume $n > 3$. Then we can choose k and m in (12.3) so that $k \simeq m \simeq n/3$ and $n - m - k \simeq n/3$. As follows from (12.4), for any nonempty finite set $A \subset \Gamma$,

$$p_{2m}^A(x, x) \leq Cn^{-\nu} \quad \text{and} \quad p_{2(n-m-k)}^A(y, y) \leq Cn^{-\nu},$$

whence, by Proposition 12.1,

$$\left| \partial_n p^A(x, y) \right| \leq Cn^{-\nu-1}.$$

By letting $A \rightarrow \Gamma$, we obtain (12.5).

If $n \leq 3$, then (12.5) follows from the trivial inequality $|\partial_n p| \leq p_n + p_{n+2}$ and the fact that (12.4) implies a similar bound for $p_n(x, y)$ (cf. (5.17) and (5.18)). □

13. Off-diagonal lower bound

An important intermediate step in proving the lower estimate (LE) is a *near-diagonal lower estimate*

$$p_n(x, y) + p_{n+1}(x, y) \geq cn^{-\alpha/\beta} \quad (NLE)$$

for all $x, y \in \Gamma$ and $n \geq 1$ such that

$$d(x, y) \leq \delta n^{1/\beta}. \quad (13.1)$$

In this section we finish the proof of lower bound (LE) in Theorem 2.1 as on the following diagram:

$$(V) + (G) \implies (FK) + (V) + (E) + (H) \implies (NLE) + (V) \implies (LE).$$

The first implication here is given by Propositions 5.5, 6.3, and 10.1, whereas the other two are proved below.

Let us recall that (DLE) refers to the lower bound

$$p_{2n}^{B(x, R)}(x, x) \geq cn^{-\alpha/\beta}, \quad \forall n \leq \varepsilon R^\beta, \quad (DLE)$$

with some small enough $\varepsilon > 0$, and (DUE) refers to the upper bound

$$p_n(x, x) \leq Cn^{-\alpha/\beta}. \quad (DUE)$$

Denote for simplicity by $(E \leq)$ the upper bound in (E) ; that is,

$$E(x, R) \leq CR^\beta, \quad \forall x \in \Gamma, R \geq 1. \quad (E \leq)$$

PROPOSITION 13.1

For any graph (Γ, μ) , we have

$$(DUE) + (DLE) + (E \leq) + (H) \implies (NLE). \quad (13.2)$$

Consequently, if (p_0) holds on (Γ, μ) , then

$$(FK) + (V) + (E) + (H) \implies (NLE). \quad (13.3)$$

Proof

Let us first show how the second claim follows from the first one. Recall that, by Proposition 5.1, $(FK) \implies (DUE)$; by Proposition 7.1, $(E) \implies (\Psi)$; and, by Proposition 9.1, $(\Psi) + (V) \implies (DLE)$. Hence, the hypotheses of (13.3) imply the hypotheses of (13.2).

To prove (13.2), fix $x \in \Gamma$, $n \geq 1$, and set

$$R = \left(\frac{n}{\varepsilon}\right)^{1/\beta} \quad (13.4)$$

for a small enough positive ε . So far we only assume that ε satisfies (DLE), but later one more upper bound on ε is imposed. Denote $A = B(x, R)$, and introduce the function

$$u(y) := p_n^A(x, y) + p_{n+1}^A(x, y).$$

By hypothesis (DLE), we have $u(x) \geq cn^{-\alpha/\beta}$. Let us show that

$$|u(x) - u(y)| \leq \frac{c}{2} n^{-\alpha/\beta} \quad (13.5)$$

for all y such that $d(x, y) \leq \delta n^{1/\beta}$, which would imply $u(y) \geq (c/2)n^{-\alpha/\beta}$ and hence prove (NLE).

The function $u(y)$ is in the class $c_0(A)$ and solves the equation $\Delta u(y) = f(y)$ where

$$f(y) := p_{n+2}^A(x, y) - p_n^A(x, y).$$

On-diagonal upper bound (DUE) implies, by Proposition 12.3,

$$\max_y |f(y)| \leq \frac{C}{n^{\alpha/\beta+1}}. \quad (13.6)$$

By (H) and Proposition 11.2, we have, for any $0 < r < R$ and for some $\sigma \in (0, 1)$,

$$\operatorname{osc}_{B(x, \sigma r)} u \leq 2 \left(\overline{E}(x, r) + \varepsilon^2 \overline{E}(x, R) \right) \max |f|. \quad (13.7)$$

By Proposition 6.1, $(E \leq)$ implies a similar upper bound for $\overline{E}$. Estimating $\max |f|$ by (13.6), we obtain, from (13.7),

$$\operatorname{osc}_{B(x, \sigma r)} u \leq C \frac{r^\beta + \varepsilon^2 R^\beta}{n^{\alpha/\beta+1}}.$$

Choosing r to satisfy $r^\beta = \varepsilon^2 R^\beta$ and substituting from (13.4) $n = \varepsilon R^\beta$, we obtain

$$\operatorname{osc}_{B(x, \sigma r)} u \leq C \frac{\varepsilon^2 R^\beta}{n^{\alpha/\beta+1}} = C \varepsilon n^{-\alpha/\beta},$$

which implies

$$\operatorname{osc}_{B(x, \sigma r)} u \leq \frac{c}{2} n^{-\alpha/\beta}, \quad (13.8)$$

provided ε is small enough.

Note that

$$\sigma r = \sigma \varepsilon^{2/\beta} R = \sigma \varepsilon^{2/\beta} \left(\frac{n}{\varepsilon} \right)^{1/\beta} = \sigma \varepsilon^{1/\beta} n^{1/\beta} = \delta n^{1/\beta},$$

where $\delta := \sigma \varepsilon^{1/\beta}$. Hence, (13.8) implies (13.5), provided $d(x, y) \leq \delta n^{1/\beta}$, which was to be proved. $\square$

The final step in proving part $(V) + (G) \implies (LE)$ of Theorem 2.1 is covered by the following statement. Denote by $(V \geq)$ the lower bound in (V) that is,

$$V(x, R) \geq cR^\alpha, \quad \forall x \in \Gamma, R \geq 1. \quad (13.9)$$

PROPOSITION 13.2

Assume that (Γ, μ) satisfies (p_0) . Then

$$(NLE) + (V \geq) \implies (LE).$$

We precede the proof with the following lemmas. Denote for simplicity

$$\tilde{P}_n = P_n + P_{n+1}, \quad (13.10)$$

where P_n is the n -convolution power of the Markov operator P . In particular, we have

$$P_n P_m = P_{n+m}. \quad (13.11)$$

We need a replacement for this property for the operator $\tilde{P}_n$, which is stated in Lemma 13.5.

LEMMA 13.3

Assume that (p_0) holds on (Γ, μ) . Then, for all integers $n \geq l \geq 1$ such that

$$n \equiv l \pmod{2}, \quad (13.12)$$

we have

$$P_l(x, y) \leq C^{n-l} P_n(x, y) \quad (13.13)$$

for all $x, y \in \Gamma$, with a constant $C = C(p_0)$.

Proof

By semigroup property (5.15), we have

$$P_{k+2}(x, y) = \sum_{z \in \Gamma} P_k(x, z) P_2(z, y) \geq P_k(x, y) P_2(y, y).$$

Using (p_0) , we obtain

$$P_2(y, y) = \sum_{z \sim y} P(y, z) P(z, y) \geq p_0 \sum_{z \sim y} P(y, z) = p_0,$$

whence $P_{k+2}(x, y) \geq p_0 P_k(x, y)$. Iterating this inequality, we obtain (13.13) with $C = p_0^{-1/2}$. $\square$

LEMMA 13.4

Assume that (Γ, μ) satisfies (p_0) . Then, for all integers $n \geq l \geq 1$ and all $x, y \in \Gamma$,

$$\tilde{P}_l(x, y) \leq C^{n-l} \tilde{P}_n(x, y), \quad (13.14)$$

where $C = C(p_0)$.

Remark 13.1

Note that no parity condition is required here in contrast to condition (13.12) of Lemma 13.3.

Proof

This is an immediate consequence of Lemma 13.3 because both $P_l(x, y)$ and $P_{l+1}(x, y)$ can be estimated from above via either $P_n(x, y)$ or $P_{n+1}(x, y)$ depending on the parity of n and l . $\square$

LEMMA 13.5

Assume that (Γ, μ) satisfies (p_0) . Then, for all $n, m \in \mathbb{N}$ and $x, y \in \Gamma$, we have the following inequality:

$$\tilde{P}_n \tilde{P}_m(x, y) \leq C \tilde{P}_{n+m+1}(x, y), \quad (13.15)$$

where $C = C(p_0)$.

Proof

Observe that, by (13.10) and (13.11),

$$\tilde{P}_n \tilde{P}_m = (P_n + P_{n+1})(P_m + P_{m+1}) = P_{n+m} + 2P_{n+m+1} + P_{n+m+2}.$$

By Lemma 13.3, $P_{n+m}(x, y) \leq C P_{n+m+2}$, whence

$$\tilde{P}_n \tilde{P}_m(x, y) \leq C(P_{n+m+1} + P_{n+m+2}) = C \tilde{P}_{n+m+1}.$$

$\square$

LEMMA 13.6

Assume that (Γ, μ) satisfies (p_0) . Then, for all $x, y \in \Gamma$ and $k, m, n \in \mathbb{N}$ such that $n \geq km + k - 1$, we have the following inequality:

$$(\tilde{P}_m)^k(x, y) \leq C^{n-km} \tilde{P}_n(x, y). \quad (13.16)$$

Proof

By induction, (13.15) implies

$$(\tilde{P}_m)^k(x, y) \leq C^{k-1} \tilde{P}_{km+k-1}(x, y).$$

From inequality (13.14) with $l = km + k - 1$, we obtain

$$\tilde{P}_{km+k-1}(x, y) \leq C^{n-km-(k-1)} \tilde{P}_n(x, y),$$

whence (13.16) follows. $\square$

Proof of Proposition 13.2

Since

$$\tilde{P}_n(x, y) = (p_n(x, y) + p_{n+1}(x, y))\mu(y),$$

(NLE) can be stated as follows:

$$\tilde{P}_n(x, y) \geq cn^{-\alpha/\beta} \mu(y) \quad \text{if } d(x, y) \leq \delta n^{1/\beta}. \quad (13.17)$$

The required (LE) takes the form

$$\tilde{P}_n(x, y) \geq cn^{-\alpha/\beta} \mu(y) \exp \left[- \left(\frac{d^\beta(x, y)}{cn} \right)^{1/(\beta-1)} \right]. \quad (13.18)$$

To prove (13.18), fix $x, y \in \Gamma$, $n \geq d(x, y)$, and consider the following cases:

Case 1: $d(x, y) \leq \delta n^{1/\beta}$,

Case 2: $\delta n^{1/\beta} < d(x, y) \leq \varepsilon n$,

Case 3: $\varepsilon n < d(x, y) \leq n$.

Here δ is the constant from (13.17) and $\varepsilon > 0$ is a small constant to be chosen later. In the first case, (13.18) coincides with (13.17). In the third case, (13.18) becomes

$$\tilde{P}_n(x, y) \geq cn^{-\alpha/\beta} \mu(y) \exp(-Cn), \quad (13.19)$$

which can be deduced directly from (p_0) . Indeed, depending on the parity of n , there is a path from x to y of length either n or $n + 1$. The $\mathbb{P}_x$ -probability that the random walk follows this path is at least $p_0^{-(n+1)}$, whence

$$\tilde{P}_n(x, y) \geq \exp(-Cn).$$

This implies (13.19), using the fact that $\mu(y) \leq C$. The latter is proved as follows. Take, in (13.17), $x \sim y$ and $n \simeq \delta^{-\beta}$. Then (13.17) implies

$$1 \geq \tilde{P}_n(x, y) \geq c\delta^\alpha \mu(y),$$

whence $\mu(y) \leq C$.

Consider the main second case. Denote $d = d(x, y)$, take a positive integer k such that

$$k \leq d, \quad (13.20)$$

Figure 9. The chain of balls $B(o_i, r)$

and define m by

$$m = \left\lfloor \frac{n}{k} \right\rfloor - 1. \quad (13.21)$$

Since $k \leq d \leq \varepsilon n$, we see that $n/k \geq \varepsilon^{-1}$ and that m is positive. Since $n \geq k(m+1)$, Lemma 13.6 applies and yields

$$C^{n-mk} \tilde{P}_n(x, y) \geq (\tilde{P}_m)^k(x, y). \quad (13.22)$$

In order to estimate $(\tilde{P}_m)^k(x, y)$, observe that there exists a sequence $o_1, o_2, \dots, o_k$ of points on Γ such that $x = o_1$, $y = o_k$, and, for all $i = 1, 2, \dots, k-1$,

$$d(o_i, o_{i+1}) \leq \left\lceil \frac{d(x, y)}{k} \right\rceil =: r \quad (13.23)$$

(see Figure 9).

Clearly, we have

$$(\tilde{P}_m)^k(x, y) \geq \sum_{z_1 \in B(o_1, r)} \cdots \sum_{z_{k-1} \in B(o_{k-1}, r)} \tilde{P}_m(x, z_1) \tilde{P}_m(z_1, z_2) \cdots \tilde{P}_m(z_{k-1}, y). \quad (13.24)$$

Assume that we have, in addition,

$$3r \leq \delta m^{1/\beta}. \quad (13.25)$$

Since $d(z_{i-1}, z_i) \leq 3r$, each $\tilde{P}_m(z_{i-1}, z_i)$ can be estimated by (13.17) as follows:

$$\tilde{P}_m(z_{i-1}, z_i) \geq cm^{-\alpha/\beta} \mu(z_i).$$

The same applies to $\tilde{P}_m(x, z_1)$ and $\tilde{P}_m(z_{k-1}, y)$. Using the lower bound of volume (13.9), we obtain, from (13.22) and (13.24),

$$\begin{aligned} C^{n-mk} \tilde{P}_n(x, y) &\geq (cm^{-\alpha/\beta})^{k-1} V(o_1, r) \cdots V(o_{k-1}, r) \mu(y) \\ &\geq c^k m^{-(\alpha/\beta)k} r^{\alpha(k-1)} \mu(y). \end{aligned}$$

Hence,

$$\tilde{P}_n(x, y) \geq c^{n-mk+k} m^{-(\alpha/\beta)k} r^{\alpha(k-1)} \geq c^k m^{-\alpha/\beta} \left(\frac{r}{m^{1/\beta}} \right)^{\alpha(k-1)}, \quad (13.26)$$

where we have used the fact that $n - mk + k \leq 3k$, which follows from (13.21).

Before we go further, let us specify the choice of k to ensure that both (13.20) and (13.25) hold. Using definitions (13.21) and (13.23) of m and r , we see that (13.25) is equivalent to

$$C \frac{d}{k} \leq \delta \left(\frac{n}{k} \right)^{1/\beta}$$

or

$$k \geq C \delta^{-\beta/(\beta-1)} \left(\frac{d^\beta}{n} \right)^{1/(\beta-1)}. \quad (13.27)$$

Let k be the minimal possible integer satisfying (13.27). By the hypothesis $d \geq \delta n^{1/\beta}$, we have

$$k \simeq \left(\frac{d^\beta}{n} \right)^{1/(\beta-1)}. \quad (13.28)$$

Condition (13.20) follows from the hypothesis $n \geq \varepsilon^{-1}d$, provided ε is small enough.

From (13.28), (13.21), and (13.25), we obtain

$$m \simeq \left(\frac{n}{d} \right)^{\beta/(\beta-1)} \quad \text{and} \quad r \simeq \left(\frac{n}{d} \right)^{1/(\beta-1)}.$$

Hence, by (13.26) and $m \leq n/k$,

$$\tilde{P}_n(x, y) \geq c^k m^{-\alpha/\beta} \geq n^{-\alpha/\beta} k^{\alpha/\beta} \exp(-Ck) \geq n^{-\alpha/\beta} \exp(-C'k).$$

Substituting here k from (13.28), we obtain (13.18). $\square$

14. Parity matters

Let us recall that (LE) contains the estimate for $p_n + p_{n+1}$ rather than for p_n . In this section we discuss to what extent it is possible to estimate p_n from below. In general, there is no lower bound for $p_n(x, y)$ for the parity reason. Indeed, on any bipartite graph, the length of any path from x to y has the same parity as $d(x, y)$. Therefore, $p_n(x, y) = 0$ if $n \not\equiv d(x, y) \pmod{2}$.

We immediately obtain the following result for bipartite graphs.

PROPOSITION 14.1

If (Γ, μ) is bipartite and satisfies (LE) , then

$$p_n(x, y) \geq cn^{-\alpha/\beta} \exp \left(- \left(\frac{d(x, y)^\beta}{cn} \right)^{1/(\beta-1)} \right) \quad (14.1)$$

for all $x, y \in \Gamma$ and $n \geq 1$ such that

$$n \geq d(x, y) \quad \text{and} \quad n \equiv d(x, y) \pmod{2}. \quad (14.2)$$

Proof

Indeed, assuming (14.2), $n + 1$ and $d(x, y)$ have different parities whence $p_{n+1}(x, y) = 0$, and (14.1) follows from (LE). $\square$

If there is enough “mixing of parity” in the graph, then one does get the lower bound regardless of the parity of n and $d(x, y)$.

PROPOSITION 14.2

Assume that graph (Γ, μ) satisfies (p_0) , (LE), and the following “mixing” condition: there is an odd positive integer n_0 such that

$$\inf_{x \in \Gamma} P_{n_0}(x, x) > 0. \quad (14.3)$$

Then lower bound (14.1) holds for all $n > n_0$ and $x, y \in \Gamma$, provided $n \geq d(x, y)$.

For example, if $n_0 = 1$, then hypothesis (14.3) means that each point $x \in \Gamma$ has a loop edge $\overline{xx}$. If $n_0 = 3$ and there are no loops, then (14.3) means that, for each point $x \in \Gamma$, there is an edge triangle $\overline{xy}, \overline{yz}, \overline{zx}$. This property holds, in particular, for the graphical Sierpiński gasket (see Figure 1).

Proof

By (9.2) we obtain, for any positive integer m ,

$$p_{2m}(x, x) \geq \frac{1}{V(x, m+1)} \left(\sum_{z \in B(x, m+1)} p_m(x, z) \mu(z) \right)^2 = \frac{1}{V(x, m+1)}.$$

Condition (p_0) and Proposition 3.1 imply $V(x, m+1) \leq C^{m+1} \mu(x)$, whence

$$P_{2m}(x, x) = p_{2m}(x, x) \mu(x) \geq C^{-m-1}.$$

Since we use this lower estimate only for the bounded range of $m \leq m_0$, we can rewrite it as

$$P_{2m}(x, x) \geq c, \quad (14.4)$$

where $c = c(m_0) > 0$.

Assuming $n > n_0$, we have, by semigroup property (5.15),

$$p_n(x, y) = \sum_{z \in \Gamma} p_{n-n_0}(x, z) P_{n_0}(z, y) \geq p_{n-n_0}(x, y) P_{n_0}(y, y) \quad (14.5)$$

and, in the same way,

$$p_n(x, y) \geq p_{n-n_0+1}P_{n_0-1}(y, y). \quad (14.6)$$

By hypothesis (14.3), we can estimate $P_{n_0}(y, y)$ from below by a positive constant. Also, $P_{n_0-1}(y, y)$ is bounded below by a constant, as in (14.4). Hence, adding up (14.5) and (14.6), we obtain

$$p_n(x, y) \geq c(p_{n-n_0}(x, y) + p_{n-n_0+1}(x, y)). \quad (14.7)$$

The right-hand side of (14.7) can be estimated from below by (LE), whence (14.1) follows. $\square$

Finally, let us show an example that explains why in general one cannot replace $p_n + p_{n+1}$ in (LE) by p_n , even assuming the parity condition $n \equiv d(x, y) \pmod{2}$.

Example 14.1

Let (Γ, μ) be $\mathbb{Z}^D$ with the standard weight $\mu_{xy} = 1$ for $x \sim y$, and let $D > 4$. We modify Γ by adding one more edge ξ of weight 1, which connects the origin $o = (0, 0, \dots, 0)$ to the point $(1, 1, 0, 0, \dots, 0)$, and we denote the new graph by (Γ', μ') .

Clearly, the volume growth and the Green kernel on (Γ', μ') are of the same order

as on (Γ, μ) ; that is,

$$V(x, r) \simeq r^D \quad \text{and} \quad g(x, y) \simeq d(x, y)^{2-D}.$$

Hence, for both graphs one has, by Theorem 2.1,

$$p_n(x, y) \leq Cn^{-D/2} \exp\left(-\frac{d^2(x, y)}{Cn}\right) \quad (14.8)$$

and a similar lower bound (LE) for $p_n + p_{n+1}$. Since $\mathbb{Z}^D$ is bipartite, we have for (Γ, μ) , by Proposition 14.1,

$$p_n(x, y) \geq cn^{-D/2} \exp\left(-\frac{d^2(x, y)}{cn}\right) \quad \text{if } n \geq d(x, y) \text{ and } n \equiv d(x, y) \pmod{2}. \quad (14.9)$$

Let us show that (Γ', μ') does not satisfy (14.9). Fix some (large) odd integer m , and consider points $x = (m, m, 0, 0, \dots, 0)$ and $y = -x$ (see Figure 10).

The distance $d(x, y)$ on Γ is equal to $4m$, whereas the distance $d'(x, y)$ on Γ' is $4m - 1$, due to the shortcut ξ . Denote $n = m^2$. Then $n \equiv d'(x, y) \pmod{2}$ and $n > d'(x, y)$. Let us estimate from above $p_n(x, y)$ on (Γ', μ') and show that it does not satisfy lower bound (14.9). Since n is odd and all odd paths from x to y have to go through the edge ξ , the strong Markov property yields

$$p_n(x, y) = \sum_{k=0}^n \mathbb{P}_x(\tau = k) p_{n-k}(o, y), \quad (14.10)$$

where τ is the first time the random walk hits the point o . If $n - k < m$, then $p_{n-k}(o, y) = 0$. If $n - k \geq m$, then we estimate $p_{n-k}(o, y)$ by (14.8) as follows:

$$p_{n-k}(o, y) \leq \frac{C}{(n-k)^{D/2}} \leq \frac{C}{m^{D/2}}.$$

Therefore, (14.10) implies

$$p_n(x, y) \leq Cm^{-D/2} \mathbb{P}_x\{\tau < \infty\}.$$

The $\mathbb{P}_x$ -probability to hit o is of the order $g(x, o) \simeq m^{2-D}$. Hence, we obtain

$$p_n(x, y) \leq Cm^{-(3D/2-2)} = Cn^{-(3D/4-1)} = o(n^{-D/2})$$

so that lower bound (14.9) cannot hold.

A more careful argument shows that, in fact, $p_n(x, y) \simeq n^{-(D-1)}$.

15. Consequences of the heat kernel estimates

Here we prove the remaining part of Theorem 2.1, as stated in the next proposition.

PROPOSITION 15.1

Assuming (p_0) , we have

$$(LE) + (UE) \implies (V) + (G).$$

Proof

The Green kernel is related to the heat kernel by

$$g(x, y) = \sum_{n=0}^{\infty} p_n(x, y). \quad (15.1)$$

Let $x \neq y$. Then $p_0(x, y) = 0$, and the upper bound (UE) for p_n implies the upper bound for g as follows:

$$g(x, y) \leq C \sum_{n=1}^{\infty} n^{-\alpha/\beta} \exp\left(-c\left(\frac{d^\beta}{n}\right)^{1/(\beta-1)}\right),$$

where $d = d(x, y)$. By estimating the sum via an integral, we obtain $g(x, y) \leq Cd^{-\gamma}$ with $\gamma = \alpha - \beta$. Similarly, one proves $g(x, y) \leq Cd^{-\gamma}$ using (LE) and the obvious consequence of (15.1):

$$g(x, y) \geq \frac{1}{2} \sum_{n=1}^{\infty} (p_n(x, y) + p_{n+1}(x, y)).$$

Let us prove the upper bound for the volume

$$V(x, R) \leq CR^\alpha \quad (V \leq)$$

for any $x \in \Gamma$ and $R \geq 1$. Indeed, for any $n \in \mathbb{N}$, we have

$$\sum_{y \in \Gamma} p_n(x, y) \mu(y) \equiv 1, \quad (15.2)$$

whence

$$\sum_{y \in B(x, R)} (p_n(x, y) + p_{n+1}(x, y)) \mu(y) \leq 2$$

and

$$V(x, R) \leq 2 \left(\inf_{y \in B(x, R)} (p_n(x, y) + p_{n+1}(x, y)) \right)^{-1}.$$

Taking $n \simeq R^\beta$ and applying (LE), we see that the inf is bounded below by $cn^{-\alpha/\beta} \simeq R^{-\alpha}$ whence $(V \leq)$ follows.

Let us prove the lower bound for the volume

$$V(x, R) \geq cR^\alpha. \quad (V \geq)$$

We first show that (UE) and $(V \leq)$ imply the following inequality:

$$\sum_{y \notin B(x, R)} p_n(x, y) \mu(y) \leq \frac{1}{2}, \quad \forall n \leq \varepsilon R^\beta, \quad (15.3)$$

provided $\varepsilon > 0$ is sufficiently small. Denoting $R_k = 2^k R$, we have

$$\begin{aligned} \sum_{y \notin B(x, R)} p_n(x, y) \mu(y) &\leq C \sum_{y \notin B(x, R)} n^{-\alpha/\beta} \exp \left[-c \left(\frac{d(x, y)^\beta}{n} \right)^{1/(\beta-1)} \right] \\ &\leq C \sum_{k=0}^{\infty} \sum_{y \in B(x, R_{k+1}) \setminus B(x, R_k)} n^{-\alpha/\beta} \exp \left[-c \left(\frac{R_k^\beta}{n} \right)^{1/(\beta-1)} \right] \\ &\leq C \sum_{k=0}^{\infty} R_k^\alpha n^{-\alpha/\beta} \exp \left[-c \left(\frac{R_k^\beta}{n} \right)^{1/(\beta-1)} \right] \\ &= C \sum_{k=0}^{\infty} \left(\frac{2^k R}{n^{1/\beta}} \right)^\alpha \exp \left[-c \left(\frac{2^k R}{n^{1/\beta}} \right)^{\beta/(\beta-1)} \right]. \end{aligned} \quad (15.4)$$

If $R/n^{1/\beta}$ is large enough, then the right-hand side of (15.4) is majorized by a geometric series, and the sum can be made arbitrarily small, in particular, smaller than $1/2$.

From (15.2) and (15.3), we conclude that

$$\sum_{y \in B(x, R)} p_n(x, y) \mu(y) \geq \frac{1}{2}, \quad (15.5)$$

whence

$$V(x, R) \geq \frac{1}{2} \left(\sup_{y \in B(x, R)} p_n(x, y) \right)^{-1}.$$

Finally, choosing $n = [\varepsilon R^\beta]$ and using the upper bound $p_n(x, y) \leq Cn^{-\alpha/\beta}$, we obtain $(V \geq)$.

This argument works only if $\varepsilon R^\beta \geq 1$. Let us now prove $(V \geq)$ for the opposite case when $\varepsilon R^\beta < 1$. To that end, define R_0 by $\varepsilon R_0^\beta = 1$. Then we have $R < R_0$. By hypothesis (p_0) and Proposition 3.1, we have $V(x, R_0) \leq C\mu(x)$. Combining with the lower bound $(V \geq)$ for $V(x, R_0)$, we obtain $\mu(x) \geq c > 0$. In particular, for any $R > 0$, we have $V(x, R) \geq c$, which implies $(V \geq)$ for the bounded range of R . $\square$

Remark 15.1

Using a similar argument, one can also show the following implication:

$$(V) + (UE) + (H) \implies (LE). \quad (15.6)$$

Indeed, as we have seen in the proof of Proposition 15.1, (UE) implies $(G \leq)$, which, together with (V) , is enough to obtain $(E \leq)$ (see Proposition 6.3). From (UE) and (V) , one obtains the diagonal lower bound $p_{2n}(x, x) \geq cn^{-\alpha/\beta}$. Indeed, from (9.2) and (15.5) with $R = Cn^{1/\beta}$, we deduce

$$p_{2n}(x, x) \geq \frac{1}{V(x, R)} \left(\sum_{y \in B(x, R)} p_n(x, y) d\mu(y) \right)^2 \geq \frac{1}{4V(x, R)} \simeq n^{-\alpha/\beta}.$$

From this estimate, one gets (DLE) (see [56]; the argument is similar to the proof of (6.6)). Also, (DUE) follows trivially from (UE) . Hence, having (DUE) , (DLE) , $(E \leq)$, and (H) , we obtain (NLE) by Proposition 13.1, and then we deduce (LE) from $(NLE) + (V)$ by Proposition 13.2.

Implication (15.6) yields that $(V) + (UE) + (H)$ is *equivalent* to either of our main conditions $(V) + (G)$ and $(UE) + (LE)$. Indeed, we have

$$(V) + (G) \implies (V) + (UE) + (H) \implies (UE) + (LE),$$

where the first implication follows by Theorem 2.1 and Proposition 10.1, and the second is the same as (15.6). We are left to close the circle by Theorem 2.1 or Proposition 15.1.

Appendix. The list of the lettered conditions

Here we provide a list of the lettered conditions frequently used in the paper. The relations among the exponents $\alpha, \beta, \gamma, \nu$ are as follows:

$$\alpha > \beta \geq 2, \quad \gamma = \alpha - \beta, \quad \text{and} \quad \nu = \alpha/\beta.$$

In all conditions, n is an arbitrary positive integer, R is an arbitrary positive real number, and x, y are arbitrary points on Γ , subject to additional restrictions if any. The constants $C, c, \delta, \varepsilon, p_0$ are positive. We have the following list:

$$V(x, R) \simeq R^\alpha, \quad \forall R \geq 1, \quad (V)$$

$$E(x, R) \simeq R^\beta, \quad \forall R \geq 1, \quad (E)$$

$$g(x, y) \simeq d(x, y)^{-\gamma}, \quad x \neq y, \quad (G)$$

$$V(x, 2R) \leq CV(x, R), \quad (D)$$

$$\overline{E}(x, R) \leq CE(x, R), \quad (\overline{E})$$

$$\lambda_1(A) \geq c\mu(A)^{-1/\nu} \quad \text{for all nonempty finite sets } A \subset \Gamma, \quad (FK)$$

$$p_n(x, x) \leq Cn^{-1/\nu}, \quad (DUE)$$

$$p_n(x, y) \leq Cn^{-\alpha/\beta} \exp \left[- \left(\frac{d(x, y)^\beta}{Cn} \right)^{1/(\beta-1)} \right], \quad (UE)$$

$$(p_n + p_{n+1})(x, y) \geq cn^{-\alpha/\beta} \exp \left[- \left(\frac{d(x, y)^\beta}{cn} \right)^{1/(\beta-1)} \right] \quad \text{if } n \geq d(x, y), \quad (LE)$$

$$p_{2n}^{B(x, R)}(x, x) \geq cn^{-\alpha/\beta} \quad \text{if } n \leq \varepsilon R^\beta, \quad (DLE)$$

$$p_n(x, y) + p_{n+1}(x, y) \geq cn^{-\alpha/\beta} \quad \text{if } d(x, y) \leq \delta n^{1/\beta}, \quad (NLE)$$

$$\Psi_n(x, R) := \mathbb{P}_x(T_{B(x, R)} \leq n) \leq C \exp \left[- \left(\frac{R^\beta}{Cn} \right)^{1/(\beta-1)} \right], \quad (\Psi)$$

$$P(x, y) \geq p_0 \quad \text{if } x \sim y, \quad (p_0)$$

$$\max_{B(x, R)} u \leq H \min_{B(x, R)} u \quad (H)$$

for any function u nonnegative in $\overline{B}(x, 2R)$ and harmonic in $B(x, 2R)$.

References

- [1] D. G. ARONSON, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 607–694, [MR 55:8553](#); *Addendum*, Ann. Scuola Norm. Sup. Pisa (3) **25** (1971), 221–228. [MR 55:8554](#) [467](#)
- [2] P. AUSCHER, *Regularity theorems and heat kernel for elliptic operators*, J. London Math. Soc. (2) **54** (1996), 284–296. [MR 97f:35034](#) [467](#)
- [3] P. AUSCHER and T. COULHON, *Gaussian lower bounds for random walks from elliptic regularity*, Ann. Inst. H. Poincaré Probab. Statist. **35** (1999), 605–630. [MR 2000m:60086](#) [467](#)

- [4] M. T. BARLOW, “Diffusions on fractals” in *Lectures on Probability Theory and Statistics (Saint-Flour, France, 1995)*, Lecture Notes in Math. **1690**, Springer, Berlin, 1998, 1–121. [MR 2000a:60148](#) [452](#), [458](#), [459](#)
- [5] M. T. BARLOW and R. F. BASS, *The construction of Brownian motion on the Sierpiński carpet*, Ann. Inst. H. Poincaré Probab. Statist. **25** (1989), 225–257. [MR 91d:60183](#) [465](#)
- [6] ———, *Transition densities for Brownian motion on the Sierpiński carpet*, Probab. Theory Related Fields **91** (1992), 307–330. [MR 93k:60203](#) [465](#), [467](#)
- [7] ———, *Brownian motion and harmonic analysis on Sierpinski carpets*, Canad. J. Math. **51** (1999), 673–744. [MR 2000i:60083](#) [453](#), [459](#), [465](#), [467](#)
- [8] ———, “Random walks on graphical Sierpinski carpets” in *Random Walks and Discrete Potential Theory (Cortona, Italy, 1997)*, Sympos. Math. **39**, Cambridge Univ. Press, Cambridge, 1999, 26–55. [MR CMP 1 802 425](#) [453](#), [458](#), [459](#), [467](#)
- [9] M. T. BARLOW, T. COULHON, and A. GRIGOR’YAN, *Manifolds and graphs with slow heat kernel decay*, to appear in Invent. Math. [458](#), [464](#)
- [10] M. T. BARLOW and E. A. PERKINS, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields **79** (1988), 543–623. [MR 89g:60241](#) [453](#)
- [11] I. BENJAMINI, I. CHAVEL, and E. A. FELDMAN, *Heat kernel lower bounds on Riemannian manifolds using the old ideas of Nash*, Proc. London Math. Soc. (3) **72** (1996), 215–240. [MR 97c:58150](#) [467](#)
- [12] A. BOUKRICHA, *Das Picard-Prinzip und verwandte Fragen bei Störung von harmonischen Räumen*, Math. Ann. **239** (1979), 247–270. [MR 81h:31018](#) [467](#), [486](#)
- [13] E. A. CARLEN, S. KUSUOKA, and D. W. STROOCK, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré Probab. Statist. **23** (1987), 245–287. [MR 88i:35066](#) [467](#), [470](#)
- [14] G. CARRON, “Inégalités isopérimétriques de Faber-Krahn et conséquences” in *Actes de la table ronde de géométrie différentielle (Luminy, France, 1992)*, Semin. Congr. **1**, Soc. Math. France, Montrouge, 1996, 205–232. [MR 97m:58198](#) [465](#), [469](#)
- [15] I. CHAVEL, *Eigenvalues in Riemannian Geometry*, Pure Appl. Math. **115**, Academic Press, Orlando, 1984. [MR 86g:58140](#) [452](#)
- [16] ———, *Isoperimetric inequalities and heat diffusion on Riemannian manifolds*, lecture notes, 1999. [452](#)
- [17] S. Y. CHENG, P. LI, and S.-T. YAU, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. **103** (1981), 1021–1063. [MR 83c:58083](#) [467](#), [491](#)
- [18] S. Y. CHENG and S.-T. YAU, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354. [MR 52:6608](#) [459](#)
- [19] F. R. K. CHUNG, *Spectral Graph Theory*, CBMS Regional Conf. Ser. in Math. **92**, Amer. Math. Soc., Providence, 1997. [MR 97k:58183](#) [460](#)
- [20] T. COULHON, *Ultracontractivity and Nash type inequalities*, J. Funct. Anal. **141** (1996), 510–539. [MR 97j:47055](#) [469](#)
- [21] T. COULHON and A. GRIGOR’YAN, *On-diagonal lower bounds for heat kernels and*

- Markov chains*, Duke Math. J. **89** (1997), 133–199. [MR 98e:58159](#) [467](#)
- [22] ———, *Random walks on graphs with regular volume growth*, Geom. Funct. Anal. **8** (1998), 656–701. [MR 99e:60153](#) [458](#), [460](#), [471](#)
- [23] T. COULHON and L. SALOFF-COSTE, *Puissances d'un opérateur régularisant*, Ann. Inst. H. Poincaré Probab. Statist. **26** (1990), 419–436. [MR 91j:43002](#) [467](#)
- [24] ———, *Minorations pour les chaînes de Markov unidimensionnelles*, Probab. Theory Related Fields **97** (1993), 423–431. [MR 95b:60085](#) [467](#)
- [25] E. B. DAVIES, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math. **92**, Cambridge Univ. Press, Cambridge, 1989. [MR 90e:35123](#) [452](#)
- [26] ———, *Pointwise bounds on the space and time derivatives of heat kernels*, J. Operator Theory **21** (1989), 367–378. [MR 90k:58214](#) [467](#)
- [27] ———, *Non-Gaussian aspects of heat kernel behaviour*, J. London Math. Soc. (2) **55** (1997), 105–125. [MR 97i:58169](#) [467](#)
- [28] T. DELMOTTE, *Parabolic Harnack inequality and estimates of Markov chains on graphs*, Rev. Mat. Iberoamericana **15** (1999), 181–232. [MR 2000b:35103](#) [453](#), [458](#)
- [29] E. B. FABES and D. W. STROOCK, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational Mech. Anal. **96** (1986), 327–338. [MR 88b:35037](#) [467](#)
- [30] S. GOLDSTEIN, “Random walks and diffusion on fractals” in *Percolation Theory and Ergodic Theory of Infinite Particle Systems* (Minneapolis, 1984/85), ed. H. Kesten, IMA Vol. Math. Appl. **8**, Springer, New York, 1987, 121–129. [MR 88g:60245](#) [453](#)
- [31] A. A. GRIGOR'YAN, *The heat equation on non-compact Riemannian manifolds* (in Russian), Mat. Sb. **182**, no. 1 (1991), 55–87; English translation in Math. USSR-Sb. **72**, no. 1 (1992), 47–77. [MR 92h:58189](#) [453](#)
- [32] ———, *Heat kernel upper bounds on a complete non-compact manifold*, Rev. Mat. Iberoamericana **10** (1994), 395–452. [MR 96b:58107](#) [465](#), [469](#)
- [33] ———, *Integral maximum principle and its applications*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 353–362. [MR 95c:35045](#) [465](#)
- [34] ———, *Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold*, J. Funct. Anal. **127** (1995), 363–389. [MR 96a:58183](#) [467](#)
- [35] ———, “Estimates of heat kernels on Riemannian manifolds” in *Spectral Theory and Geometry* (Edinburgh, 1998), ed. B. Davies and Yu. Safarov, London Math. Soc. Lecture Note Ser. **273**, Cambridge Univ. Press, Cambridge, 1999, 140–225. [MR 2001b:58040](#) [452](#)
- [36] B. M. HAMBLY and T. KUMAGAI, *Transition density estimates for diffusion processes on post critically finite self-similar fractals*, Proc. London Math. Soc. (3) **78** (1999), 431–458. [MR 99m:60118](#) [453](#)
- [37] W. HEBISCH and L. SALOFF-COSTE, *Gaussian estimates for Markov chains and random walks on groups*, Ann. Probab. **21** (1993), 673–709. [MR 94m:60144](#) [452](#), [458](#)
- [38] ———, *On the relation between elliptic and parabolic Harnack inequalities*, preprint, 2000. [467](#), [468](#)

- [39] O. D. JONES, *Transition probabilities for the simple random walk on the Sierpiński graph*, Stochastic Process. Appl. **61** (1996), 45–69. [MR 97b:60115](#) 453, 458
- [40] J. KIGAMI, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Amer. Math. Soc. **335** (1993), 721–755. [MR 93d:39008](#) 453
- [41] ———, *Harmonic calculus on limits of networks and its application to dendrites*, J. Funct. Anal. **128** (1995), 48–86. [MR 96e:60130](#) 453
- [42] J. KIGAMI and M. L. LAPIDUS, *Weyl’s problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Comm. Math. Phys. **158** (1993), 93–125. [MR 94m:58225](#) 453
- [43] N. V. KRYLOV and M. V. SAFONOV, *A certain property of solutions of parabolic equations with measurable coefficients* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **44**, no. 1 (1980), 161–175, 239; English translation in Math. USSR-Izv. **16** (1981), 151–164. [MR 83c:35059](#) 467
- [44] T. KUMAGAI, *Estimates of transition densities for Brownian motion on nested fractals*, Probab. Theory Related Fields **96** (1993), 205–224. [MR 94e:60068](#) 453
- [45] S. KUSUOKA and Z. X. YIN [X. Y. ZHOU], *Dirichlet forms on fractals: Poincaré constant and resistance*, Probab. Theory Related Fields **93** (1992), 169–196. [MR 94e:60069](#) 453
- [46] E. M. LANDIS, *The Second Order Equations of Elliptic and Parabolic Type* (in Russian), Izdat. “Nauka,” Moscow, 1971, [MR 47:9044](#); English translation in Transl. Math. Monogr. **171**, Amer. Math. Soc., Providence, 1998. [MR 98k:35034](#) 467
- [47] P. LI and S.-T. YAU, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201. [MR 87f:58156](#) 452
- [48] F. LUST-PIQUARD, *Lower bounds on $\|K^n\|_{1 \rightarrow \infty}$ for some contractions K of $L^2(\mu)$, with applications to Markov operators*, Math. Ann. **303** (1995), 699–712. [MR 96m:47055](#) 458, 467
- [49] J. MOSER, *On Harnack’s theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. [MR 28:2356](#) 467
- [50] ———, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134, [MR 28:2357](#); *Correction*, Comm. Pure Appl. Math. **20** (1967), 231–236. [MR 34:3121](#) 467
- [51] J. NASH, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954. [MR 20:6592](#) 470
- [52] F. O. PORPER and S. D. ĖĖDEL’MAN, *Two-side estimates of fundamental solutions of second-order parabolic equations and some applications* (in Russian), Uspekhi Mat. Nauk **39**, no. 3 (1984), 107–156; English translation in Russian Math. Surveys **39**, no. 3 (1984), 119–179. [MR 86b:35078](#) 452
- [53] L. SALOFF-COSTE, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices **1992**, 27–38. [MR 93d:58158](#) 453
- [54] ———, *Isoperimetric inequalities and decay of iterated kernels for almost-transitive Markov chains*, Combin. Probab. Comput. **4** (1995), 419–442. [MR 97c:60171](#) 458
- [55] R. SCHOEN and S.-T. YAU, *Lectures on Differential Geometry*, Conf. Proc. Lecture

- Notes Geom. Topology **1**, International Press, Cambridge, Mass., 1994.
 MR 97d:53001 452
- [56] D. W. STROOCK, “Estimates on the heat kernel for second order divergence form operators” in *Probability Theory (Singapore, 1989)*, ed. L. H. Y. Chen, K. P. Choi, K. Hu, and J. H. Lou, de Gruyter, Berlin, 1992, 29–44. MR 93m:35092 467, 505
- [57] K-T. STURM, *Analysis on local Dirichlet spaces, III: The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), 273–297. MR 97k:31010 454
- [58] M. TAKEDA, *On a martingale method for symmetric diffusion processes and its applications*, Osaka J. Math. **26** (1989), 605–623. MR 91d:60193 465
- [59] A. TELCS, *Random walks on graphs, electric networks and fractals*, Probab. Theory Related Fields **82** (1989), 435–449. MR 90h:60065 453, 458, 465, 475, 477, 478
- [60] ———, *Spectra of graphs and fractal dimensions, I*, Probab. Theory Related Fields **85** (1990), 489–497. MR 91k:60075 478
- [61] ———, *Spectra of graphs and fractal dimensions, II*, J. Theoret. Probab. **8** (1995), 77–96. MR 96d:60107 478
- [62] ———, *Transition probability estimates for reversible Markov chains*, Electron. Comm. Probab. **5** (2000), 29–37, <http://www.math.washington.edu/~ejpecp/> MR 2001b:60088 458
- [63] N. TH. VAROPOULOS, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal. **63** (1985), 240–260. MR 87a:31011 469
- [64] ———, *Analysis on nilpotent groups*, J. Funct. Anal. **66** (1986), 406–431. MR 88h:22014 452
- [65] N. TH. VAROPOULOS, L. SALOFF-COSTE, and T. COULHON, *Analysis and Geometry on Groups*, Cambridge Tracts in Math. **100**, Cambridge Univ. Press, Cambridge, 1992. MR 95f:43008 452
- [66] W. WOESS, *Random walks on infinite graphs and groups—a survey on selected topics*, Bull. London Math. Soc. **26** (1994), 1–60. MR 94i:60081 452, 459
- [67] X. Y. ZHOU, *Resistance dimension, random walk dimension and fractal dimension*, J. Theoret. Probab. **6** (1993), 635–652. MR 95d:60119 453

Girgor'yan

Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, United Kingdom; a.grigoryan@ic.ac.uk

Telcs

International Management Center, Graduate School of Business, Budapest, Zrinyi u. 14, Budapest H-1051, Hungary; h197tel@ella.hu