Approximate Numerical Method To Predict Lubrication with Generalised Newtonian Fluids

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Abstract

In this paper an approach is proposed to account for Generalised Newtonian lubrication flow in a lubricated bearing geometry. The approximate approach solves the lubricant viscosity across the lubricated gap, the coefficients in the equation describing the viscosity across the gap are then solved for iteratively. This permits an approximation of the lubrication equation to be arrived at which considers the Poiseuille components of the flow as well as the Couette terms. The reduction in error of around 30 – 40% is observed in the bearing pressure field for the cases considered here.

1 Introduction

A wide range of flows have been successfully analysed by means of the lubrication approximation [1, 2, 3] and Reynolds' equation[4]. In recent years some attention has turned to the effect of non-Newtonian fluid characteristics on fluid flow, with interest in shear thinning fluid lubrication being one such area [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The methods used have included the use of an average shear rate between the lubricated surfaces on which to base the viscosity, as well as the full analytical solutions of the flow problem as obtained by Johnson & Mangkoesoebroto [8] (see also Das [11]) for shear thinning fluids obeying the power law [16, 17]. Such analytical solutions have been restricted to two dimensional flow. The application of Reynolds equation in which the local fluid viscosity has been calculated as a function of the average shear rate across the fluid film results in a viscosity being purely a function of the Couette component of the flow, with that derived from the Poiseuille component neglected. As shown by Hewson et al [17] large discrepancies between the predicted pressure field using the averaged shear based viscosity and the full analytical solution of the lubrication equation can arise. The importance of the inclusion of the Poiseuille flow component derived viscosity becomes even greater for purely pressure driven flow.

2 Calculation Procedure

In subsection 2.1 the general solution method used to solve the shear thinning flow in a bearing is detailed, this analysis involves assuming a variation in viscosity between the two lubricated surfaces and the selection of a shear thinning rheological model. In subsection 2.2 the method outlined in subsection 2.1 is followed using a specific viscosity function and the

2.1 Generalised Solution Method

The general calculation procedure used to solve the approximate lubrication equation is similar to that used to derived Reynolds equation. The difference being that a viscosity function that describes the variation in viscosity between the lubricated surfaces is assumed. The viscosity function is then solved using the appropriate boundary condition based on the viscosity as determined by the chosen viscosity model. For an assumed viscosity function the velocity in the x and y directions are obtained by solving,

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial z} \left[\eta(z) \frac{\partial u}{\partial z} \right] \quad \text{and} \quad \frac{\partial p}{\partial y} = \frac{\partial}{\partial z} \left[\eta(z) \frac{\partial v}{\partial z} \right] \quad (1)$$

for the application of appropriate velocity boundary conditions at $z = 0$ and $z = h$. It is therefore important for an integratable function to be obtained for $\frac{1}{\eta(z)}$. The velocities are integrated across the lubricated gap to obtain the flow rates, q_x and q_y , in the x and y directions respectively.

$$q_x = \int_0^h u dz \quad \text{and} \quad q_y = \int_0^h v dz \quad (2)$$

By applying conservation of volume in the x and y directions the following is obtained:

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \quad (3)$$

$$\frac{\partial}{\partial x} \left[A(z) \frac{\partial p}{\partial x} + B(z) \right] + \frac{\partial}{\partial y} \left[C(z) \frac{\partial p}{\partial y} + D(z) \right] = 0, \quad (4)$$

where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are functions of z . The following iterative solution method is then used.

1. By defining an initial $\eta(z)$ the linear equation (4) can be solved numerically for the pressure field.
2. The solution is then used to provide a more accurate approximation for the viscosity function. This is done by obtaining the viscosity and the appropriate number of viscosity derivatives at $z = 0$ and $z = h$ based on the shear rates at these locations.
3. Based on the revised viscosity function an updated pressure field is obtained, and this is repeated until convergence of the pressure distribution.

2.2 Shear Thinning Analysis

Considering the four parameter Carreau-Yassuda viscosity model. The viscosity function used in the subsequent analysis is:

$$\eta(z) = \frac{1}{az^3 + bz^2 + cz + d}, \quad (5)$$

where a , b , c and d are dependent only on the x - y location. By applying the boundary conditions $u = 0$ at $z = 0$, $u = 1$ at $z = h$ and $v = 0$ at $z = 0, h$ and solving the flow equation for volume flow rate which can then be applied to equation (3), yielding,

$$\begin{aligned} \frac{\partial}{\partial x} \left[\frac{-6h^3 \left(h^6 a^2 + \frac{10}{3} h^5 ab + \left(\frac{5}{2} b^2 + \frac{15}{2} ca \right) h^4 + (30da + 10bc) h^3 + \left(\frac{110}{3} db + \frac{25}{3} c^2 \right) h^2 + 50dch + 50d^2 \right)}{900h^3 a + 1200h^2 b + 1800hc + 3600d} \frac{\partial p}{\partial x} \right. \\ \left. - 6 \frac{h \left(-30h^3 a - 300d - 100hc - 50h^2 b \right)}{900h^3 a + 1200h^2 b + 1800hc + 3600d} \right] + \\ \frac{\partial}{\partial y} \left[\frac{-6h^3 \left(h^6 a^2 + \frac{10}{3} h^5 ab + \left(\frac{5}{2} b^2 + \frac{15}{2} ca \right) h^4 + (30da + 10bc) h^3 + \left(\frac{110}{3} db + \frac{25}{3} c^2 \right) h^2 + 50dch + 50d^2 \right)}{900h^3 a + 1200h^2 b + 1800hc + 3600d} \frac{\partial p}{\partial y} \right] = 0. \end{aligned} \quad (6)$$

By assuming initial values of $a = 0$, $b = 0$, $c = 0$ and $d = 1$ which in this case is the solution for Newtonian fluids, equation (6) is solved for, with appropriate pressure boundary conditions. From this solution the first and second derivatives of velocities u_1 and u_2 with respect to z are determined analytically. From these velocity gradients the shear rate can be determined:

$$\gamma = \sqrt{\left(\frac{du}{dz} \right)^2 + \left(\frac{dv}{dz} \right)^2}, \quad (7)$$

in addition the first derivate with respect to z can also be found:

$$\frac{d\dot{\gamma}}{dz} = \frac{\left(\frac{du}{dz} \frac{d^2 u}{dz^2} + \frac{dv}{dz} \frac{d^2 v}{dz^2} \right)}{\sqrt{\left(\frac{du}{dz} \right)^2 + \left(\frac{dv}{dz} \right)^2}} \quad (8)$$

Using the dimensionless Carreau-Yassuda viscosity model:

$$\eta = 1 + (\eta_{ratio} - 1) (1 + |\lambda \dot{\gamma}|^\alpha)^{\frac{n-1}{\alpha}}, \quad (9)$$

where η_{ratio} is the ratio of zero shear viscosity to infinite shear viscosity and λ , α and n are the constants of the rheological model. The first derivative of viscosity with z was also determined:

$$\frac{d}{dz} \eta = \frac{(\eta_{ratio} - 1) (1 + |\lambda \dot{\gamma}|^\alpha)^{\frac{n-1}{\alpha}} (n-1) |-\lambda \dot{\gamma}|^\alpha \frac{d\dot{\gamma}}{dz}}{\dot{\gamma} (1 + |\lambda \dot{\gamma}|^\alpha)}. \quad (10)$$

It is therefore possible to obtain both the viscosity and the rate of change of viscosity with z at $z = 0$ and $z = h$ (further derivatives can also be found). It is from these four values that the unknowns in the viscosity function are obtained, by simultaneously solving the viscosity function and its first derivative with z at $z = 0$ and $z = h$. Equation (6) is then solved iteratively with new values of a , b , c and d being calculated for each subsequent step.

α	2
λ	3
η_{ratio}	$1 \rightarrow 2$
n	$0 \rightarrow 1$

Table 1: Carreau-Yassuda Fluid Test Case Properties Examined

3 Test Case

The test case examined is that of a three dimensional converging gap, as is typically found in a slider bearing geometry.

Figure 1: Geometry of test case examined in the current study.

The boundary conditions applied are $p = 0$ at $x = 0$, $x = 100$ and $y = 0$. The symmetry of the problem along the $y = 25$ plane allows a symmetry boundary conditions $\left(\frac{\partial p}{\partial y}\right)$ to be applied along the $y = 25$ plane and is used in all subsequent analysis. A range of variables were examined, these are given in table 1.

3.1 Lubrication Model

The pressure equation (equation (6)) was implemented in Comsol Multiphysics 3.2b¹. The finite element mesh comprised of 100 elements in the x -direction and 25 elements in the y direction. Grid independence was confirmed by running test cases at 4 times the element density (by doubling the number of elements in each direction). Convergence of the iterative process was assumed when the maximum change in pressure field from one iteration to the next was less than 10^{-6} times the maximum pressure predicted. Convergence was stable for all cases examined (it should be noted that when the power law viscosity model was imple-

¹a multiphysics finite element package from COMSOL AB, Chalmers Teknikpark, SE-412 88, Göteborg, Sweden.

mented rather than the Carreau-Yassuda model convergence was not always assured, most likely reason for this being the unbounded viscosity when zero shear stress was encountered at $z = 0$ or $z = h$ leading to numerical instability. By using a viscosity model with a zero shear rate limit this problem was avoided (in addition to more accurately capturing the low shear rate dependence).

3.2 Three Dimensional Simulation

A full 3D analysis of the flow domain for Carreau-Yassuda fluids was solved the viscous flow equations for the primitives u, v, w and p for Carreau-Yassuda fluids. A block mesh was used (second order in velocity and first order in pressure) with 50 elements in the x , 18 elements in the y direction and 6 elements in the z . With this mesh density the results were found to be mesh independent, with test cases calculated with 60, 20 and 8 elements in the x, y and z directions respectively considered.

4 Results

One test of the modelling method is to compare the pressure field predictions of the approximate method with those of the three-dimensional finite element simulation. In addition to this a comparison was also made with the pressure field from an adapted form of Reynolds equation where the local fluid viscosity is a function of the average shear rate (details on this can be found in references [9, 10, 12, 13]).

(a) 4th order viscosity function (equation 5)

(b) Average shear based viscosity ($\dot{\gamma} = \frac{u}{h}$)

Figure 2: Percentage error in pressure field of the analytical simulations of the load bearing capabilities of the test geometry (figure 1); $\alpha = 2$, $\lambda = 3$, $n = 0.5$ and $\eta_{ratio} = 2$.

Figure 2 shows the error between the three dimensional finite element simulation and the approximate methods. This error is the difference between the pressures given by the approximate method and the finite element method divided by the maximum pressure predicted by the finite element method and then multiplied by 100. It is clear that the error in pressure field predicted by the 4th order viscosity function method is around of 1/3 of that predicted by the average shear rate viscosity function. This result is reflected in the load bearing capabilities of the bearing, obtained by integrating the pressure over the bearing surface. The percentage error in the load bearing capabilities of the bearing as predicted by the two approximate methods is given in figure 3.

Figure 3: Percentage error of the analytical simulations of the load bearing capabilities of the test geometry (figure 1); $\alpha = 2$, $\lambda = 3$.

It can be seen by comparing figures 3a and 3b that the predictions from the 4th order viscosity function are significantly better than those from the average shear based viscosity results, which gives percentage errors 2-3 times greater than the 4th order viscosity function based solution. An increase in the error is also observed with a decrease in n which gives a measure of the rate of transition from the low to high shear viscosity with shear rate.

5 Conclusion

In conclusion the improvement in the predicted pressure distributions when compared to the commonly used average shear rate viscosity method is good and the method used to approximate for the viscosity appears promising. There is further potential to examine the choice of viscosity function with a view to improve the accuracy of the computational method. Other functions may be found to provide more accurate representations of the viscosity, including functions that generate viscosities bounded by the low and high shear rate viscosity limits, something missing from the current model and which may give rise to

viscosities inconsistent with the viscosity model employed (in this case the Carreau-Yassuda model).

References

- [1] H.M. Martin. The lubrication of gear teeth. *Engineering*, 102:119, 1916.
- [2] E.K. Gatcombe. Lubrication characteristics of involute spur gears. *Transactions of the ASME*, 67:177, 1945.
- [3] W. H. Banks and C. C. Mill. Some observations on the behaviour of liquids between rotating rollers. *Proceedings of the Royal Society of London A*, 223:414, 1954.
- [4] O. Reynolds. On the theory of lubrication and its application to mr beauchamp tower's experiment, including an experimental determination of the viscosity of olive oil. *Philosophical Transactions of the Royal Society of London*, 177:157–234, 1886.
- [5] Dowson D. A generalised reynolds equation for fluid film lubrication. *International Journal of Mechanical Science*, 4:159–170, 1962.
- [6] D. J. Coyle, C. W. Macosko, and L. E. Scriven. Film-splitting flows of shear-thinning liquids in forward roll coating. *AIChE Journal*, 33(5):741–746, 1987.
- [7] D. J. Coyle, C. W. Macosko, and L. E. Scriven. Reverse roll coating of non-Newtonian liquids. *Journal of Rheology*, 34(5):615–636, 1990.
- [8] M. W. Johnson and S. Mangkoesoebroto. Analysis of lubrication theory for the power law fluid. *Journal of Tribology-Transactions of the Asme*, 115(1):71–77, 1993.
- [9] R. I. Taylor, M. A. Brown, D. M. Thompson, and J. C. Bell. The influence of lubricant rheology on friction in the piston ring pack. *SAE paper 941981*, 1994.
- [10] T. Tian, V. W. Wong, and J. B. Heywood. A piston ring-pack film thickness and friction model for multigrade oils and rough surfaces. *SAE paper 962032*, 1996.
- [11] N. C. Das. A study of optimum load capacity of slider bearings lubricated with power law fluids. *Tribology International*, 32(8):435–441, 1999.
- [12] R. I. Taylor. The inclusion of lubricant shear thinning in the short bearing approximation. *Proceedings of the Institution of Mechanical Engineers Part J- Journal of Engineering Tribology*, 213(J1):35–46, 1999.
- [13] H. Hirani, K. Athre, and S. Biswas. Lubricant shear thinning analysis of engine journal bearings. *Tribology Transactions*, 44(1):125–131, 2001.
- [14] T. G. Myers. Application of non-newtonian models to thin film flow. *Physical Review E*, 72(6), 2005.
- [15] Y. Peiran and W. Shizhu. A generalised Reynolds-equatin for non-Newtonina thermal elastohydrodynamics lubrication, journal = Journal of Tribology-Transactions of the Asme, volume = 112, number = 4, pages = 631-636, year = 1990.

- [16] J. G. Oldroyd. On the formulation of rheological equations of state. *Proceedings of the Royal Society of London Series A – Mathematical and Physical Sciences*, 200(1063):523–541, 1950.
- [17] R. W. Hewson, N. Kapur, and P. H. Gaskell. A model for film-forming with newtonian and shear-thinning fluids. *Journal of Non-Newtonian Fluid Mechanics*, 162(1-3):21–28, 2009.