Comparative assessment of finite-element and strut and tie based design methods for deep beams

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This paper considers the design and analysis of a series of one and two-span deep beams that have been tested previously. The beams are assessed with non-linear finite-element analysis and two alternative strut and tie models. The reinforcement is subsequently designed to resist between 60 and 80% of the measured failure loads using a semi-automated finite-element based procedure and the two strut and tie models. The semi-automated design procedure is implemented in NonOpt, which is a Fortran program that works in conjunction with the finite-element program Diana. The reinforcement is designed to resist stresses calculated in a non-linear finite-element analysis. The case studies with NonOpt examine the influence of basing the initial reinforcement design on either a linear or non-linear finite-element analysis. The influence of tension stiffening is also considered. The least reinforcement weight is obtained when the initial design is based on a non-linear analysis without tension stiffening.

Notation

A_i	surface area of reinforcement band i	n_d	number of divisions in reinforcement mesh used in NonOpt
A_{sw}	total area of effective stirrups in shear span under consideration	n_r	number of mesh refinements in NonOpt
a	shear span between centrelines of loading and support plates	T_1	tensile force resisted by tie T_1 at midspan
a_v	clear shear span between inner edges of loading and support plates	T_2	tensile force resisted by tie T_2 over internal support
b	beam width	T_d	longitudinal component of force resisted by direct strut I
c	distance from bottom of beam to centroid of bottom flexural reinforcement	T_i	longitudinal component of force resisted by strut II
d	effective depth	T'_i	longitudinal component of force resisted by strut III
f'_c	mean concrete cylinder strength	T_{si}	total tensile force resisted by effective stirrups in internal shear span
f_{ck}	characteristic concrete cylinder strength	V	shear force
f_{cn}	concrete strength in node	w_b	width of direct strut at its bottom end
f_{csb}	concrete strength at bottom end of direct strut	w_t	width of direct strut at its top end
f_{cst}	concrete strength at top end of direct strut	β	fraction of longitudinal force $T_d + T'_i$ transferred to bottom node by direct strut I
f_y	yield strength of reinforcement	ϕ'	inclination of strut III to the horizontal
h	beam depth	γ_c	material factor of safety for concrete
L	span between centreline of adjacent supports	ρ_i	reinforcement ratio of reinforcement band i
l_{be}	length of external support	ρ_l	longitudinal reinforcement ratio
l_{bi}	length of internal support	ρ_t	transverse reinforcement ratio
l_t	length of loaded area	λ	proportion of shear carried by direct strut I in strut and tie model
M	total number reinforcement bands	ν'	strength reduction factor for cracked concrete in shear
		θ	inclination of direct strut I to the horizontal

Introduction

The authors have previously described a novel automated procedure for the design of reinforced concrete structures such as deep beams and shear walls that are reinforced with orthogonal grids of reinforcement (Amini Najafian and Vollum, 2013a). The procedure finds the areas of reinforcement required to resist stresses calculated in a non-linear finite-element analysis (NLFEA) of the complete structure. The equations of the modified compression field theory (MCFT) (Collins *et al.*, 2008) are used in both the reinforcement design and the NLFEA, but any similar constitutive model could be used. The user is able to specify design constraints including maximum permissible values for the principal compressive and tensile strains in the concrete and the maximum reinforcement stress at cracks. The design constraints are expressed in terms of safety factors, which are calculated for each design constraint as the ratio of the permissible to actual value. Practical reinforcement arrangements are achieved by grouping the finite elements into horizontal (HB) and vertical bands (VB) as shown in Figure 1. The horizontal reinforcement ratio ρ_l is constant within any given HB. Likewise, the vertical reinforcement ratio ρ_t is constant within any given VB. The elements at the intersections of the HB and VB bands are referred to as VHB bunches.

The design procedure is implemented in NonOpt (Amini Najafian, 2011), which is a Fortran program that works in conjunction with the finite-element program Diana (TNO Diana, 2007). NonOpt includes three alternative strategies for determining the reinforcement within each band that are depicted A, B and C. Each procedure is iterative owing to the dependency of the stresses calculated in the NLFEA on the reinforcement ratios determined in the previous step. This paper considers design strategy A (Amini Najafian and Vollum, 2013b), which reinforces all the elements in each reinforcement band with the greatest of the individual reinforcement ratios required in each of its elements. By way of illustration, strategy A deals with the

linkage between the VHB in HB1 of Figure 1 by setting its reinforcement ratio ρ_l equal to the greatest of the ratios ρ_l required in VHB1, VHB3 and VHB5. Strategy B is designed for computational efficiency and has similarities with strategy C (Amini Najafian and Vollum, 2013a), which minimises either the sum of the reinforcement ratios or the overall reinforcement weight, which is proportional to $\sum_{i=1}^M A_i \rho_i$ in which A_i is the surface area of reinforcement band i with reinforcement ratio ρ_i and M is the total number reinforcement bands. Strategy B (Amini Najafian and Vollum, 2013c) minimises the sum of the reinforcement ratios in up to four linked VHB, which are selected on the basis of their relative safety factors. It gives comparable results to strategy C, but is computationally more efficient.

Strategy A uses a direct search method to find the minimum area of reinforcement that satisfies the design constraints in each VHB. The first step is to generate an equally spaced mesh with n_d divisions in the $\rho_l - \rho_t$ plane in which ρ_l and ρ_t vary between $\rho_{\min}$ and $\rho_{\max}$. The solution procedure involves moving sequentially through the mesh along parallel lines on which $\rho_l + \rho_t = \text{const}$ is shown in Figure 2 until a solution is found that satisfies the design constraints at every Gauss point in the VHB bunch. At this point, the reinforcement mesh is either refined as shown in Figure 2 or the remaining points on the current parallel line are checked to determine whether there are any additional solutions. The mesh is refined between the parallel line passing through the solution and the adjacent parallel line with $\delta\rho$ less total reinforcement, where the incremental step in the mesh is

$$\delta\rho = \frac{\rho_{\max} - \rho_{\min}}{n_d}$$

The reinforcement combination with the greatest safety factor is adopted in cases where multiple solutions exist with the same value of $\rho_l + \rho_t$. The reinforcement is updated after each design

Figure 1. Definition of VB, HB and VHB bunches in a finite-element model

Figure 2. Mesh generation and refinement

iteration prior to being fed back to Diana (TNO Diana, 2007) as illustrated in the flowchart of Figure 3. A full description of strategy A can be found in Amini Najafian and Vollum (2013b). Strategies B and C use similar direct search strategies to strategy A but the direct search solution procedure becomes M -dimensional when M reinforcement ratios are involved in the minimisation (Amini Najafian and Vollum, 2013a, 2013c). Strategy A is considered in this paper as it has previously been shown to give only slightly greater reinforcement weights than strategies B and C with considerably less computation (Amini Najafian and Vollum, 2013a, 2013c).

The authors' method is intended as an alternative to strut and tie modelling, which can be very demanding for complex structures unless automated. The main attraction of the strut and tie method (STM) is that the loads are resisted by an idealised truss, which makes it straightforward to check that equilibrium is satisfied and the reinforcement is adequately anchored. The method has the drawback that it is often difficult to determine an appropriate geometry for the idealised truss as there is no unique solution. Furthermore, the adopted truss may have insufficient ductility for the assumed design forces to develop and crack widths may be excessive at the serviceability limit state. Park *et al.* (2010) have addressed this by developing an integrated design framework in which NLFEA is used to assess the behaviour of structures

designed using STM. The authors' design procedure also uses NLFEA to assess structural behaviour but it eliminates the need to develop a STM by designing the reinforcement to resist stresses calculated in a NLFEA. It has the advantage that the same constitutive relationships are used in the reinforcement design as in the subsequent assessment with NLFEA, which enables explicit performance-based design constraints, such as crack widths, to be considered in the reinforcement design. The authors' method also allows for the contribution of the minimal reinforcement that codes require for crack control unlike STMs, which can be overly conservative, as noted by Quintero-Febres *et al.* (2006).

This paper presents a series of case studies in which NonOpt is used to design the reinforcement required in a number of deep beams tested by Rogowsky *et al.* (1983, 1986). The resulting reinforcement areas are compared with those used in the tested beams as well as those given by two alternative STMs.

Analysis of deep beams tested by Rogowsky and co-workers

Rogowsky *et al.* (1983, 1986) tested a series of 24 deep beams of which 17 were continuous over two spans. The ratio of the shear span to effective depth (a/d) was taken as 1.0, 1.5, 2.0 and 2.5 where a is the distance from the centreline of the load to the centreline of the support. The beams were reinforced with (a) no web reinforcement, (b) minimum vertical stirrups, (c) minimum horizontal web reinforcement, (d) minimum vertical stirrups and minimum horizontal web reinforcement, (e) maximum vertical stirrups and (f) maximum horizontal web reinforcement. Beams BM2/1.5, BM5/1.5, BM8/1.5, BM2/2.0, BM3/2.0 and BM5/2.0 are considered in this study of which beams BM2/ (north span) and BM8/1.5 were reinforced with horizontal and vertical web reinforcement. The BM2 beams had a single span whereas the other beams were continuous over two spans. The numbering system for the beams consists of three parts, thus in the notation BM2/1.5, BM2 denotes the beam and reinforcement type and the symbol 1.5 indicates that the a/d ratio was 1.5. The beam dimensions and reinforcement arrangement for each series are illustrated in Figure 4. Further details of the specimens and their material properties are given in Tables 1 and 2, which should be read in conjunction with Figure 4. The maximum aggregate size was 10 mm. The beams were initially loaded to failure after which the shear span in which failure occurred was strengthened with external stirrups before the beams were reloaded to failure. The continuous beams failed in the internal shear spans. Table 3 lists the shear strengths measured in each of the two loading stages. All the beams failed in shear but some of the beams failed in a ductile fashion as noted in Table 3 as a result of the flexural reinforcement yielding. The other beams failed suddenly in shear without warning.

Analysis with NLFEA

The accuracy of the results given by NonOpt is dependent on the reliability of the NLFEA procedure used in Diana. The latter was

Figure 3. Flowchart for design procedure in NonOpt

Figure 4. Geometrical and reinforcement details of (a) single-span and (b) two-span beams of Rogowsky *et al.* (1983, 1986) (stub column bars omitted for clarity)

Series	A	B	C	D
/1.5	750	300	300	600
/2.0	800	200	300	500

Note: Refer to Figure 4 for the definition of dimensions A to D, which are in mm.

Table 1. Geometric details of beams of Rogowsky *et al.* (1983, 1986)

assessed by using it to determine the failure loads of beams BM2/1.5, BM5/1.5, BM8/1.5, BM2/2.0, BM3/2.0 and BM5/2.0, which were analysed using the finite-element meshes shown in Figure 5(a) to (d). The dimensions of the reinforcement bands in Figure 5(a) to (d) correspond to the reinforcement arrangements used in the tests. The same reinforcement bands are used

subsequently in the designs with NonOpt. The concrete is modelled with the CQ16M element in DIANA, which is an eight-noded quadrilateral isoparametric plane stress element based on quadratic interpolation and Gauss integration. Four Gauss points are used in each element. The reinforcement is modelled with embedded grid elements, which are perfectly bonded to the surrounding concrete. The stress–strain response of the reinforcement is modelled using a bilinear idealisation of the measured stress–strain response. The concrete compressive strength is taken as its measured value and the strain at the peak stress is taken as $\epsilon'_c = -0.002$ in accordance with the recommendations of Eurocode 2 (BSI, 2004). The concrete compressive behaviour is modelled in accordance with the recommendations of Collins and Porasz (1989) as described below

1.
$$\sigma_d = -\beta f'_c \frac{m(\epsilon_d/\epsilon'_c)}{(m-1) + (\epsilon_d/\epsilon'_c)^{mk}}$$

Specimen	f'_c : MPa	Top steel			Bottom steel			Web steel ^a	
		Number of bars — area per bar: mm ²	$A_s f_y$ per bar: kN	d : mm	Number of bars — area per bar: mm ²	$A_s f_y$ per bar: kN	d : mm	Number of stirrups	Number of horizontal bars
BM2/1.5N	42.4	2–28.3	16.2	580	6–200	91	535	5 ^d	4
BM2/1.5S								—	4
BM2/2.0N	43.2	2–28.3	16.2	480	4–200	91	455	4	4
BM2/2.0S								—	4
BM3/2.0	42.5	4–200 ^b	91	445	4–200	91	445	4	
		2–100	48	455 ^c	2–100	48			—
BM5/1.5	39.6	6–200 ^b	91	535	4–200	91	545	16 ^e	
				555 ^c	2–100	48			—
BM5/2.0	41.1	4–200 ^b	91	445	4–200	91	445	16 ^e	
		2–100	48	455 ^c	2–100	48			—
BM8/1.5	37.2	6–200 ^b	91	535	4–200	91	545	5 ^d	4
				555 ^c	2–100	46			

Note: ^aAll web reinforcement was 6 mm deformed bars with $A_s f_y = 16.2$ kN per bar; ^bOwing to cut-off of bars within internal shear span only four bars with area 200 mm² were reported as being fully effective; ^cEffective depth of fully effective bars; ^dfour stirrups assumed to be effective in STM1; ^e14 stirrups assumed to be effective in STM1.

Table 2. Details of beams of Rogowsky *et al.* (1983, 1986)

Specimen	Measured shear strength: kN	V_{pred}/V_{test}					P_{flex}^g/P_{test}
		NLFEA		STM1		STM2	Flexure ^f
		No tension stiffening	Tension stiffening	Eurocode 2	MCFT	Eurocode 2	
BM2/1.5N ^a	348	0.64	0.86	0.87	0.94	0.77	0.93
BM2/1.5S	226	0.64	—	1.18	1.11	1.18	1.43
BM2/2.0N ^a	204	0.75	1.02	0.88	1.03	0.72	0.86
BM2/2.0S ^a	185	0.46	—	0.79	0.75	0.79	0.95
BM3/2.0	261 ^b –277 ^c	0.87–0.84	1.29–1.21	1.05–1.11	0.93–0.87	0.82–0.77	1.37–1.32
BM5/1.5 ^{ae}	565 ^b –566 ^c	0.66	0.88	0.90	1.04	0.75	1.06–1.03
BM5/2.0 ^{ae}	453 ^b –456 ^c	0.68–0.67	0.92	1.00 ^d	1.00 ^d	0.67	0.99–0.96
BM8/1.5	339 ^b –382 ^c	0.83–0.74	1.27–1.13	1.16–1.03	0.97–0.86	0.91–0.80	1.49–1.34
Mean		0.71	1.06	1.00	0.95	0.80	NA
Standard deviation		0.12	0.17	0.13	0.10	0.14	NA
Covariance		0.17	0.16	0.13	0.10	0.18	NA

Note: ^aDuctile shear failure; ^bShear force in critical interior shear span at initial failure; ^cShear force in other interior shear span at its failure subsequent to strengthening of shear span in which failure initially occurred; ^d14 stirrups assumed to yield in inner shear span at failure; ^eAll top bars assumed to yield over internal support; ^fPredicted to be critical when less than V_{pred}/V_{test} ; ^gCritical sections assumed to form at faces of stub columns.

Table 3. Comparison of measured and predicted shear forces in critical shear span of beams of Rogowsky *et al.* (1983, 1986) at failure

Figure 5. Details of finite-element mesh and bunch arrangement used in analysis/design of beams (Rogowsky *et al.*, 1983, 1986): (a) BM2/1.5, (b) BM2/2.0, (c) BM5/1.5 and BM8/1.5 and (d) BM5/2.0 (*continued*)

(d)

Figure 5. (continued)

in which

$$2. \quad m = 0.8 + \frac{f'_c}{17} \text{ (MPa)}$$

$$3. \quad k = 1; \quad \epsilon'_c < \epsilon_d < 0$$

$$4. \quad k = 0.67 + \frac{\beta f'_c}{62} \text{ (MPa); } \quad \epsilon_d \leq \epsilon'_c$$

$$5. \quad \beta = \frac{1}{1 + k_c} \leq 1$$

$$6. \quad k_c = 0.27 \left(-\frac{\epsilon_r}{\epsilon'_c} - 0.37 \right)$$

$$7. \quad \sigma_r = \frac{f_{cr}}{1 + (200\epsilon_r)^{0.5}}$$

in which the concrete tensile strength f_{cr} is taken as $f_{cr} = 0.33(f'_c)^{0.5}$. In cases where tension stiffening is neglected, f_{cr} is taken as zero.

The measured and predicted failure loads of each beam are compared in Table 3, which gives failure loads calculated with and without tension stiffening. The NLFEA is seen to give conservative estimates of the measured shear strengths when tension stiffening is omitted, but the failure loads are overestimated for some beams when it is included. This is because Diana, unlike NonOpt, does not check that the reinforcement is sufficient to transfer the tensile stress in the concrete through cracks as required for equilibrium (Collins *et al.*, 2008). In addition, NonOpt limits the concrete compressive strain in the reinforcement design to a maximum of the strain at the peak stress ϵ'_c . The general conservatism of NonOpt was validated by using it to check whether or not the reinforcement provided in the tests is sufficient to maintain equilibrium at the measured failure load of the beams in Table 3 whose strength is overestimated. These analyses showed that the provided reinforcement is insufficient, which is significant since it indicates that the design approach adopted within NonOpt is safe. In reality, the tensile stress would reduce within the concrete between the cracks when the reinforcement yields, allowing equilibrium to be satisfied at the cracks, but this is not modelled in DIANA. This

Following cracking, the principal concrete tensile stress is calculated as follows in accordance with the recommendations of Vecchio and Collins (1986)

does not affect the safety of the design procedure as the reinforcement is designed to maintain equilibrium at cracks.

Strut and tie analysis

The failure loads of the Rogowsky *et al.* (1983, 1986) beams were also estimated using the strut and tie models depicted STM1 and STM2 in Figures 6 and 7, respectively. STM1 is drawn for a continuous beam but can be readily adapted for the analysis of simply supported beams by removing the tie T_2 over the internal support. It is an extension of the model developed by Sagaseta and Vollum (2010) for simply supported beams whereas STM2 is based on the recommendations of ACI 318 (ACI, 2011) for simply supported deep beams. The horizontal web reinforcement is neglected in STM1 but this is not considered significant as Rogowsky *et al.* (1983, 1986) concluded that it had little if any influence on the measured shear resistance.

STM1

The load is assumed to be transferred from the loading plate to the supports through a direct strut (strut I) acting in parallel with a truss system (strut II–stirrups–strut III) as shown in Figure 6. The bearing stress under the loading and supporting plates is limited to $\nu' f_{ck}/\gamma_c$ at compression nodes without ties and $0.85\nu' f_{ck}/\gamma_c$ at compression nodes with ties as required by Eurocode 2 (where $\nu' = (1 - f_{ck}/250)$, f_{ck} is the characteristic concrete cylinder strength, and γ_c is the material factor of safety for concrete which Eurocode 2 takes as 1.5). The stress distribution is assumed to be uniformly distributed across the width of the node faces and non-hydrostatic. The strength of struts I and II is reduced by cracking and transverse tensile strains induced by the stirrups, which are assumed to be effective within the central three-quarters of the shear span as required by Eurocode 2. Strut III is fan-shaped like strut II, but the concrete in this region is essentially uncracked. Flexural continuity over the internal support has the effect of increasing the shear force in the internal shear spans above that in a comparable simply supported beam. It also makes the STM statically indeterminate unless the top

Figure 7. STM2 for simply supported beam

flexural reinforcement yields in tension. Analysis of the test results of Rogowsky *et al.* (1983, 1986) shows that prior to yield, the hogging moment was typically between 60 and 70% of the moment of 0.1875PL given by elastic beam analysis. Equations are presented for shear failure in the internal shear span as this was critical in the tests of Rogowsky *et al.* (1983, 1986) but equations for the shear resistance of the external spans can be derived similarly. The failure load P is defined in terms of the tensile strength T_{si} of the effective stirrups in the internal shear span as follows

$$8. \quad V_{int} = \frac{T_{si}}{(1 - \lambda)}$$

in which $T_{si} = A_{sw} f_y$ where A_{sw} is the total area of effective stirrups and λ is the proportion of V_{int} resisted by the direct strut. Stirrups are assumed to yield at failure provided that $\lambda > 0$.

The horizontal component of force in the concrete at the centre of the node over the internal support $C = T_d + T'_i$, where T_d and T'_i equal the longitudinal components of force in struts I and III, respectively

Figure 6. Details of STM1 for internal shear span of continuous beam

$$9. \quad T'_i = T_{si} \cot \phi'$$

$$10. \quad T_d = \beta T$$

$$11. \quad T_d = \frac{\lambda T_{si}}{1 - \lambda} \cot \theta = \frac{\beta T'_i}{1 - \beta}$$

$$12. \quad \beta = \frac{K + K \cot^2 \theta - 0.5 \lambda l_{bi}}{x_i \cot \theta}$$

$$13. \quad K = \frac{\lambda}{(1 - \lambda)} \frac{T_{si}}{b f_{sb}}$$

where $f_{sb} = \min(f_{csb}, f_{cst} w_t / w_b)$ is the stress in the direct strut at its bottom node when the strut fails owing to concrete crushing at either its top or bottom node, which are of widths w_t and w_b , respectively. The coefficients f_{csb} and f_{cst} denote the concrete strengths at the bottom and top ends of the direct strut. The coefficient β and the angles θ and ϕ' are defined in Figure 6.

The concrete strengths f_{csb} and f_{cst} are calculated in accordance with the recommendations of both Collins *et al.* (2008) and Eurocode 2 (BSI, 2004). Collins *et al.* (2008) define the concrete strength in the direct strut as

$$14. \quad f_{cs} = \phi f_{ck} / (0.8 + 170 \varepsilon_1)$$

where ϕ is a capacity reduction factor. In cases where the end of the strut is crossed by a tie

$$15. \quad \varepsilon_1 = \varepsilon_L + (\varepsilon_L + 0.002) \cot^2 \theta$$

where ε_L is the strain in the tie that was calculated in terms of $T = T_1 - T_i$ at the internal support, in which T_i is the longitudinal component of force resisted by strut II, and $T = T_2 - T'_i$ at the concentrated load. Canadian Standard CSA A23.3 (CSA, 2004) defines f_{cs} as $0.85 \phi f_{ck}$ at the end of a strut that is not crossed by a tension tie.

Eurocode 2 defines the design concrete strength of struts in cracked compression zones as $f_{cs} = 0.6 v' f_{ck} / \gamma_c$ where $v' = (1 - f_{ck} / 250)$. Although not explicitly stated in Eurocode 2, this strength is applied at both ends of the direct strut irrespective of whether the adjoining node is crossed by a tie as otherwise the shear strength of beams with $a_v/d > 1.0$ can be progressively

overestimated with increasing a_v/d . The overestimate in strength depends on the strain in the flexural reinforcement and the dimensions of the bearing plates (Sagasetta and Vollum, 2010).

The widths of the direct strut at its top and bottom ends, w_t and w_b , respectively, are given by

$$16. \quad w_t = \lambda l_{ti} \sin \theta + x_{td} \cos \theta$$

$$17. \quad w_b = 0.5 \lambda l_{bi} \sin \theta + \beta x_{bi} \cos \theta$$

in which $x_{td} = (T_d / b f_{cnt})$, $l_{ti} = (V_{int} / P) l_t$, $P = 2[V_{int} - (M_{sup} / L)]$ and $L = 2a_v + 0.25 l_{bi} + l_t + 0.5 l_{be}$ (see Figure 6 for definition of dimensions).

The depth of the node over the internal support x_{bi} is calculated from axial equilibrium as follows

$$18. \quad x_{bi} = \frac{T_d + T'_i}{b f_{cnb}} \geq 2c$$

where the stress $f_{cnb} \leq 0.85 v' f_{ck} / \gamma_c$.

The depth of the node under the central load x_t is given by

$$19. \quad x_t = \frac{T_d + T_i}{b f_{cnt}} \geq 2d'$$

where the stress $f_{cnt} \leq k v' f_{ck} / \gamma_c$ in which $k = 1.0$ for compression nodes without ties and 0.85 for compression nodes with ties. d' is the distance from the top of the beam to the centroid of the tie T_2 , which resists a tensile force equal to

$$20. \quad T_2 = [M_{sup} + f_{cnb} b x_{bi} (0.5 x_{bi} - c)] / (h - c - d')$$

where $M_{sup} = 0.1875 \delta PL \leq M_y$ in which L is the distance between the centrelines of the supports and δ is the ratio of the support moment to its elastic value of $0.1875 PL$.

The angles θ and ϕ' , which define the orientation of struts I and III in Figure 6 are given by

$$21. \quad \cot \theta = \frac{a_v + 0.25 \lambda l_{bi} + 0.5 \lambda l_{ti}}{h - 0.5 \beta x_{bi} - 0.5 T_d / b f_{cnt}}$$

$$22. \quad \cot \phi' = \frac{0.5a_v + 0.25l_{bi}(1 + \lambda)}{h - 0.5x_{bi}(1 + \beta) - d'}$$

The ultimate load is taken as the lowest value corresponding to either flexural failure, crushing of the direct strut at either end or bearing failure. Limiting the force in the direct strut of the internal shear span to $C_d = \min(f_{csb}w_b, f_{cst}w_t)b$ and imposing vertical equilibrium at the bottom node leads to

$$23. \quad \lambda V_{int} = \frac{\lambda}{1 - \lambda} T_{si} = C_d \sin \theta$$

The shear resistance V_{int} is readily calculated using the following iterative procedure

1. Estimate λ and θ .
2. Calculate T'_i , β and T_d with Equations 9 to 13, respectively.
3. Calculate $\cot \phi'$ with Equation 22.
4. Calculate new values for $\cot \theta$ and λ as follows

$$24. \quad \cot \theta_{i+1} = \frac{\beta}{\lambda} \left(\frac{1 - \lambda}{1 - \beta} \right) \cot \phi'$$

$$25. \quad \lambda_{i+1} = \frac{[h - 0.5\beta x_i - (0.5T_d/(bf_{cnt}))] \cot \theta - a_v}{0.25l_b + 0.5l_{ti}}$$

5. Return to step 2 and repeat steps 2 to 4 until $\cot \theta$ and λ converge to the specified tolerance in successive iterations.
6. Calculate the shear resistance V_{int} with equation (23) and hence P from moment equilibrium.

STM2

The load is assumed to be transferred to the supports through the direct struts that are modelled as bottle stress fields as shown in Figure 7. The strength of the direct struts is calculated in terms of their width at each end, which is calculated with Equations 16 and 17 with λ and $\beta = 1$. It follows that the shear resistance of the inner shear span is given by

$$26. \quad V_{int} = \min(f_{sb}w_b, f_{st}w_t)b \sin \theta$$

Following the recommendations of Eurocode 2 (BSI, 2004) for full discontinuity regions, the maximum allowable stress at the ends of the direct strut can be calculated in terms of the area of transverse reinforcement as follows

$$27. \quad f_s = \frac{2T}{wb \left(1 - \frac{0.7w}{H} \right)} \geq 0.6(1 - f_{ck}/250)f_{ck}$$

in which w is the width of the strut at its top or bottom node as appropriate, T is the force provided by the reinforcement normal to the centreline of the strut, b is the member thickness and H is the length of the strut between its loaded ends at the nodes. The force T is given by

$$28. \quad T = \sum A_{sh}f_{yh} \sin \theta + \sum A_{sv}f_{yv} \cos \theta$$

where $\sum A_{sh}$ and $\sum A_{sv}$ are the total areas of horizontal and vertical web reinforcement crossing the direct strut and f_{yh} and f_{yv} are the yield strengths of the horizontal and vertical web reinforcement, respectively. θ is the angle of inclination of the direct strut to the horizontal, which is given by Equation 21 with λ and β equal to 1.

STM analysis results

STM1 and STM2 were used to estimate the strengths of beams BM2/1.5, BM5/1.5, BM8/1.5, BM2/2.0, BM3/2.0 and BM5/2.0 of Rogowsky *et al.* (1983, 1986). Beams BM2/1.5 and BM2/2.0 were single span, whereas the other beams were continuous over two spans. The tensile force in the top flexural reinforcement of the continuous beams is indeterminate and needs to be assumed unless strain compatibility is accounted for in the STM. From the point of view of analysis, it is conservative to assume that the bending moment at the internal support equals the lesser of the elastic value of 0.1875PL calculated using beam theory, or the moment of resistance as this maximises the shear force in the inner span, which is critical for the tested beams. However, the test results show that this approach underestimates the shear force in the external spans and that the ratio between the reactions at the inner and outer supports is better estimated if the support moment is assumed to equal 70% of its elastic value prior to yield of the flexural reinforcement. Therefore, the support moment was assumed to be 70% of its elastic value, but not greater than the yield moment.

The results of the analyses are given in Table 3 along with the predicted flexural failure loads that were calculated neglecting strain hardening and the contribution of the horizontal web reinforcement as in the STMs. The flexural failure loads in Table 3 are greater than given by the STM neglecting shear failure, as flexural hinges were assumed to develop at the faces of the stub columns. Table 3 shows that STM1 gives reasonable predictions of the measured shear strengths with the MCFT predictions being the safest and most consistent. STM2 underestimates the contribution of the shear reinforcement, which it predicts only to increase the shear resistance of the B5 beams, which were very heavily reinforced with stirrups. Consequently, STM2 predicts the

same shear resistance for the north and south shear spans of the B2/beams unlike STM1, which correctly predicts the stirrups in the north shear span to increase shear resistance. Statistics are presented for the shear strength predictions of each STM even when greater than the predicted flexural strength as the ultimate strength of the flexural reinforcement was around 1.7 times the yield strength. Furthermore, all the beams failed in shear.

Design of deep beams

This section considers the design of the reinforcement in the deep beams considered in the previous section with NonOpt, STM1 and STM2. The beams were designed for the maximum possible load achievable with NonOpt without tension stiffening, which varied between 60 and 78% of the measured failure load as shown in Tables 4 and 5. The resulting design loads were typically between 90 and 95% of the NLFEA failure loads calculated without tension stiffening. The case studies with NonOpt examine the influences of the initial reinforcement arrangement and tension stiffening on the final reinforcement design.

The measured concrete strengths were used and the material factors of safety were taken as 1.0 in all the designs. The strength of the direct strut was calculated in accordance with the recommendations of the MCFT in STM1 as this approach gives the most consistent estimates of shear resistance. The same stress-strain relationship was assumed for all the reinforcement unlike the tested beams where the yield strength varied with bar diameter. The reinforcement was modelled bilinearly in NonOpt with an elastic modulus of $E_s = 200$ GPa, yield strength $f_y = 500$ MPa and post-yield tangent modulus $E'_s = 0.842$ GPa. The reinforcement stress at cracks was limited to a maximum of $f_{sler,per} = f_{sler,per} = 536$ MPa, which is the stress at 90% of the assumed strain at the maximum force. In some cases with no tension stiffening, it was also necessary to restrict the mean strain in the reinforcement to 0.0025 to obtain convergence. The minimum reinforcement ratio was taken as $\rho_{min} = 0.002$ and the

maximum ratio as $\rho_{max} = 0.04$. The same concrete constitutive relationships were used as in the analyses described in the previous section. The maximum permissible principal compressive strain in the concrete was taken as $|\epsilon_{d,per}| = 0.0017$ but other limits could be used. No limits were applied to the maximum principal tensile strain in concrete at the ultimate limit state. The crack spacing was calculated automatically in the maximum stress computations assuming that the concrete cover and bar spacing were 25 mm and 200 mm, respectively. The number of reinforcement mesh divisions in NonOpt was taken as $n_d = 10$ with one level of mesh refinement ($n_r = 1$). The initial reinforcement ratios were taken as ρ_{min} if the initial reinforcement design was based on a linear finite-element analysis and as the ratios in the tested beams if the initial design was based on a NLFEA.

Design results

Tables 4 and 5 summarise the reinforcement areas and weights given by NonOpt and the STM for the simply supported and continuous beams, respectively. The reinforcement weights given in Tables 4 and 5 include the reinforcement in the stub columns within which the reinforcement ratios are assumed to be minimal in the STM designs. The calculated reinforcement weights are significantly lower than the weights in the tested beams for all the designs. This is because the design loads were between 60 and 80% of the measured failure loads. The reinforcement weight increases disproportionately with increasing design loads, as the contribution of the direct strut reduces as the area of stirrups increases.

NonOpt is seen to give the lowest reinforcement weights for the designs denoted NLTS = 0 in which the initial reinforcement design is based on stresses calculated in a NLFEA with no tension stiffening. The reinforcement weights from the NLTS = 0 designs are similar to those given by STM1 but the distribution of reinforcement is somewhat different. For example, the NonOpt analyses with NLTS = 0 give significantly greater areas for the stirrups in the inner shear spans of the continuous beams than

Specimen	Reinforcement areas: mm ²	Test	STM1	STM2	NLTS = 0	NLTS = 1	LinTS = 0
BM2/1.5	Bottom tie T1	1200	765	765	792	595	585
	Stirrups	283	300 ^a	300 ^a	471	813	927
	Horizontal web	113	172 ^a	172 ^a	172 [†]	564	793
$P_d = 420$ kN $= 0.6P_u$	Weight: kg	52.5	21.6	21.6	24.0	31.1	35.4
	Bottom tie T1	800	674	674	693	501	419
	Stirrups	226	320 ^a	320 ^a	320 ^a	806	806
$P_d = 306$ kN $= 0.75P_u$	Horizontal web	113	148 ^a	148 ^a	148 ^a	570	654
	Weight: kg	39.9	19.0	19.0	19.3	27.7	31.8

Notes: NL: initial reinforcement design based on NLFEA with reinforcement ratios in tested beams, Lin: initial reinforcement design based on linear finite-element analysis with minimal reinforcement, TS = 0 no tension stiffening, TS = 1 tension stiffening included, ^a $\rho = 0.002$.

Table 4. Comparison of reinforcement areas and weights given by NonOpt and STM for simply supported beams of Rogowsky et al. (1983, 1986)

Specimen	Reinforcement areas: mm ²	Test	STM1	STM2	NLTS = 0	NLTS = 1	LinTS = 1
BM5/1.5 $P_d = 510 \text{ kN}$ $= 0.59 P_u$	Bottom tie T1	1000	668	668	640	412	353
	Top tie T2	1200	562	562	506	342	309
	Stirrups out ^a	905	300 ^c	300 ^a	414	642	642
	Stirrups in ^b	905	300 ^c	611	813	414	300 ^c
	Horizontal web	0	161 ^c	161 ^a	192	591	744
	Weight: kg	130.8	58.1	61.0	58.1	60.2	61.8
BM5/2.0 $P_d = 400 \text{ kN}$ $= 0.59 P_u$	Bottom tie T1	1000	678	678	629	451	412
	Top tie T2	1000	551	551	470	323	300
	Stirrups out ^a	905	320 ^c	320 ^a	381	746	746
	Stirrups in ^b	905	320 ^c	598	746	442	442
	Horizontal web	0	141 ^c	141 ^a	141 ^c	491	679
	Weight: kg	114.4	52.2	54.4	50.5	52.8	56.7
BM8/1.5 $P_d = 420 \text{ kN}$ $= 0.78 P_u$	Bottom tie T1	1000	550	550	521	300	273
	Top tie T2	1200	463	463	401	266	263
	Stirrups out ^a	283	300 ^c	300 ^a	300 ^c	300 ^c	300 ^c
	Stirrups in ^b	283	300 ^c	300 ^a	585	357	300 ^c
	Horizontal web	113	161 ^c	161 ^a	161 ^a	529	652
	Weight: kg	123.8	51.0	51.0	46.9	47.2	50.0

Notes: NL: initial reinforcement design based on NLFEA with reinforcement ratios in tested beams, Lin: initial reinforcement design based on linear finite-element analysis with minimal reinforcement, TS = 0 no tension stiffening, TS = 1 tension stiffening included, ^aouter shear span, ^binner shear span, ^c $\rho = 0.002$.

Table 5. Comparison of reinforcement areas and weights given by NonOpt and STM for continuous beams of Rogowsky *et al.* (1983, 1986)

STM1. The areas of horizontal web reinforcement in Tables 4 and 5 are, however, equal or close to the minimum allowable of $\rho_{\min} = 0.002$, which is provided in the beams designed with STM1 although not required for strength. The NLTS = 0 designs for the continuous beams give slightly lower reinforcement areas for ties T_1 and T_2 than the STM as a result of the contribution of the horizontal web reinforcement to flexural resistance being neglected in the STM. Nevertheless, the similarity between the areas of the tie T_2 given by NonOpt with NLTS = 0 and the STM confirms that it is reasonable to take the design support moment as 70% of the elastic value in the STM.

The areas of horizontal and vertical web reinforcement shown in Table 4 for STM2 were chosen to minimise the total weight of web reinforcement. Consequently, minimum horizontal web reinforcement was provided in conjunction with designed stirrups. Table 4 shows that STM2 requires significantly greater areas of stirrups than STM1 in cases where design shear reinforcement is required. STM2 becomes progressively less efficient as the design load is increased. For example, STM2 requires the same stirrups in the inner shear span of beam BM5/2.0 as STM1 for a design load of 325 kN but 60% more for a design load of 510 kN.

The reinforcement weight increases, and its distribution changes significantly, when tension stiffening is included in NonOpt

irrespective of the initial reinforcement arrangement. This arises owing to the change in the axial stress distribution over the depth of the section, which results in a large increase in the area of horizontal web reinforcement due to the reduction in the flexural lever arm. The weight of reinforcement is typically greatest when the initial reinforcement design is based on the elastic stress field irrespective of whether or not tension stiffening is included. Figures 8 to 10 show the reinforcement stresses and principal concrete strains at the design load for the NLTS = 0 designs of beams BM2/1.5, BM5/1.5 and BM5/2.0. The figures show that the maximum reinforcement stresses are close to 500 MPa, as assumed in the STM. Furthermore, the maximum principal compressive strain is within the design limit of 0.0017 except in a few elements at the corners of the loaded areas, which were neglected in the finite-element designs.

As previously discussed, the initial reinforcement ratios were taken as those in the tested beams in the designs in which the initial finite-element analysis was non-linear. In practice, the initial reinforcement design could be based on simplified considerations of flexure and shear. The stirrups could for example be designed for a reduced shear force of βV where $\beta = a_v/2d$ as the load is applied to the top of the beam within a distance of $a_v < 2d$ of the support (Eurocode 2, BSI, 2004). The main requirement is that sufficient reinforcement should be provided

Figure 8. Assessment of BM2/1.5 design NLTs = 0 at design load: (a) reinforcement stress s_{xx} , (b) reinforcement stress s_{yy} and (c) concrete minimum principal strain (compression negative)

Figure 9. Assessment of BM5/1.5 design NLTS = 0 at design load: (a) reinforcement stress s_{xx} , (b) reinforcement stress s_{yy} and (c) concrete maximum principal strain (tension positive)

Figure 10. Assessment of BM5/2.0 design NLTS = 0 at design load: (a) reinforcement stress s_{xx} , (b) reinforcement stress s_{yy} and (c) concrete minimum principal strain (compression negative)

Specimen	Reinforcement areas: mm ²	1 [Test]		2		3	
		Initial	Final	Initial	Final	Initial	Final
BM2/1.5 $P_d = 420 \text{ kN}$ $= 0.6P_u$	Bottom tie T1	1200	792	909	783	909	773
	Stirrups	283	471	525	528	900	927
	Horizontal web	113	172 ^a	206	172 ^a	206	172 ^a
	Weight: kg	52	24.0	28.3	24.3	31.8	28.1

Table 6. Influence of initial reinforcement arrangement on final reinforcement areas and weights for beam BM2/1.5 of Rogowsky *et al.* (1983, 1986) for $P_d = 420 \text{ kN}$ with NLTS = 0, ^a $\rho = 0.002$

for the initial NLFEA to converge but the area of stirrups should be kept to a minimum to maximise the contribution of the direct strut, which reduces as the area of stirrups is increased. This is illustrated in Table 6, which shows the influence of various initial reinforcement arrangements on the final arrangement for beam B2/1.5 with NLTS = 0. The table shows that the final area of stirrups increases in proportion with the initial area owing to the reduced contribution of the direct strut.

Conclusions

This paper considers the use of NLFEA and STM for the design and assessment of simply supported and continuous deep beams. The paper shows that STM is a reliable technique for evaluating the shear resistance of short span beams particularly if the strength of the direct struts is calculated in accordance with the recommendations of the MCFT. Two STM are considered. STM1 consists of a direct strut acting in parallel with a truss system whereas STM2 models the direct strut as a bottle stress field. STM1 gives significantly more accurate predictions of shear strength than STM2, which underestimates the contribution of the web reinforcement. The STM are shown to be in some respects more reliable than the NLFEA, which is complicated by the presence of high strain concentrations at the corners of the loaded areas, which can cause premature divergence.

The beams were designed with both STM and NonOpt to resist close to the failure load of the tested beams determined with NLFEA without tension stiffening. It is shown that the reinforcement areas calculated with NonOpt depend on whether or not the initial reinforcement design is based on the results of a linear or non-linear finite-element analysis. The reinforcement weight and distribution also depends on whether or not tension stiffening is included in the analysis. The least weight of reinforcement was obtained when the initial reinforcement design was based on NLFEA with no tension stiffening (NLTS = 0). The area of horizontal web reinforcement increased significantly when tension stiffening was included owing to the change in the shape of the axial stress distribution over the depth of the beam. The reinforcement weights and distributions given by STM1 and NonOpt with NLTS = 0 compare favourably, which gives confidence in

the NonOpt designs. It should be noted that STM1 took many man-hours to develop and implement. NonOpt has the advantage that it eliminates the need to develop a STM and deals readily with multiple point loads as well as variations in the positions of the loads. Design constraints can also be specified for the reinforcement design as discussed earlier.

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