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Simplified Theory of Axial Propagation Through Structurally Chiral Media

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Abstract Simple design formulae are derived for the co-, contra- and cross- handed reflection and transmission coefficients for light incident axially to a structurally chiral medium surrounded by dissimilar isotropic dielectrics. The results obtained agree well with exact calculations over the full range of achieved experimental parameters of practically manufactured structurally chiral media. The formulae are used to illustrate a number of design strategies.

Keywords: Structurally chiral media, coupled wave theory, chiral photonic media

1 Introduction

Young and Kowal [1] were the first to notice that deposition of a thin film on a rotating substrate gave rise to media that had large optical activity. Subsequently, it was shown [2] that the polarization properties of such films are derived from two key properties: glancing angle film deposition gives rise to form birefringence of a nominally isotropic dielectric; and substrate rotation promotes the growth of helical columns giving rise to a structural helicity of the dielectric tensor. In many respects the resultant artificial chiral media resemble cholesteric liquid crystals [3], and therefore they inherit some of their applications including display technology for colour filtering [4]. However, being solid, their polarization properties are hard-wired and they are therefore better described as a one-dimensional photonic crystal, a material system around which many complementary applications have been envisaged. Infiltration by gases, liquids and liquid crystals modifies the optical properties that can be used for sensing [5], for example. A rigid structure can also support controlled defects, that give rise to localized photonic modes and ultra-narrow band transmission [5, 6, 7, 8]. Since structurally chiral media (SCM) respond asymmetrically with respect to circularly polarized light, many applications based on polarization filtering and coupling have been explored, even in commercially developed fibre-based devices [9]. The nano-structuring of such artificial chiral media engages with the natural helicity of the photon, and applications to quantum information processing have recently been speculated [10]. SCM have also been shown to mimic closely the polarization properties of insects in the natural world [11] such as the scarab beetle.

The basic optical response of a structurally chiral medium (SCM) can be described as follows. Circularly polarized light incident along the helical axis that is co-handed with the structure will see a periodic variation in the refractive index. If the optical wavelength coincides with half the

helical pitch then a strong reflection occurs as a result of the Bragg resonance. Contra-handed polarization, on the other hand, will propagate as if in an isotropic medium with the average refractive index. Whilst this physics is simply stated, there has been considerable effort over the last decade to provide a complete theoretical description (see [12] and references therein). One point of especial note is that the electromagnetic problem of axial propagation through an SCM is one that can be solved *exactly*. Although the solution is fully analytic, it cannot be stated in terms of simple functions and the underlying Bragg physics is not transparent in this formalism. The author has therefore sought to achieve a simpler description based on coupled wave theory, known to well-describe scalar Bragg gratings [13]. This programme has been quite successful. In [14, 15] we grafted scalar coupled wave theory onto a description of SCM and obtained expressions for the reflections and transmissions. In [16] the theory was extended to embrace optical activity and circular dichroism. Reference [17] introduced the so-called energy flow model, in which chirality preserving reflections within the medium were combined with chirality reversing interface reflections. This yielded much simpler formulae, although only the lowest order propagation paths were considered. More detailed formulae were obtained in [18] that properly accounted for interfaces, although the explicit formulae were still somewhat cumbersome and, moreover, it was assumed that the surrounding isotropic media were of identical refractive index, and that the chiral medium was lossless. Reference [19] combined the ideas of the energy flow model with interface reflections, to produce accurate simple expressions for axial reflection from a semi-infinite chiral medium. This extended earlier work on a chiral half-space that was restricted to the Bragg regime [20]. There remained the challenge of producing simple intuitive formulae that account for *all* possible propagation paths (i.e. in the spirit of the energy flow model), but are applicable to the most general situation of an absorptive SCM of finite thickness surrounded by dissimilar isotropic dielectrics. This

paper addresses this challenge, and produces, we believe, the simplest possible analytic formulae describing axial propagation in structurally chiral media.

The paper is organized as follows: in Section 2 we derive the relevant Helmholtz equation for a SCM, elucidating its precise connection with coupled wave theory in Section 3. Then in Section 4 we show how distributed chiral reflection/transmission is combined with localized interfacial transmission and reflection by self consistently enumerating all possible propagation pathways. Simple expressions are there derived for all co-, cross- and contra-handed reflection and transmission coefficients. The alternative methodology based on the exact, but cumbersome, solution to Maxwell's equations is briefly reviewed in Section 5 before comparing results obtained by the two methods in Section 6. Various consistency checks and special cases are applied in Section 7 which also discusses design strategies that arise directly from examination of the simplified formulae. In Section 8 we conclude.

2 Chiral Film Geometry and the Helmholtz Equation for Axial Propagation

A structurally chiral dielectric medium (see Figure 1) is characterised by a relative permittivity tensor given by

$$\boldsymbol{\epsilon} = \mathbf{R} \cdot \boldsymbol{\chi} \cdot \boldsymbol{\epsilon}_{\text{ref}} \cdot \boldsymbol{\chi}^{-1} \cdot \mathbf{R}^{-1}, \quad (1)$$

where in Cartesian coordinates $\boldsymbol{\epsilon}_{\text{ref}} = \text{diag}(\epsilon_a, \epsilon_b, \epsilon_c)$ and

$$\mathbf{R} = \begin{bmatrix} \cos pz & -\sin pz & 0 \\ \sin pz & \cos pz & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\chi} = \begin{bmatrix} \cos \chi & 0 & -\sin \chi \\ 0 & 1 & 0 \\ \sin \chi & 0 & \cos \chi \end{bmatrix}. \quad (2)$$

In (2), χ tilts the principal axes of the reference dielectric tensor ϵ in the x - z plane, whilst $\mathbf{R}$ rotates these axes periodically about the z -axis with spatial period $L_p = 2\pi/p$. Since the eigenvectors of ϵ twist periodically about the z -axis in the sense of a right-handed helix, the chiral medium is right-handed in this case. Figure 1 depicts a right-handed SCM for which $\chi = 0$. The mathematical description seeks solutions to the frequency domain Maxwell curl relations for plane waves propagating along the z -axis and oscillating at $e^{-i\omega t}$:

$$\hat{\mathbf{z}} \frac{d}{dz} \times \mathbf{E} = i\omega\mu_0 \mathbf{H}, \quad (3)$$

$$\hat{\mathbf{z}} \frac{d}{dz} \times \mathbf{H} = -i\omega\epsilon_0 \epsilon \cdot \mathbf{E}. \quad (4)$$

Now since from Eq. (4) $\mathbf{D} = \epsilon_0 \epsilon \cdot \mathbf{E}$ lies in the x - y plane, the x and y components of $\epsilon_0^{-1} \epsilon^{-1} \cdot \mathbf{D}$ depend only on D_x and D_y , which can therefore be written as $\epsilon_0^{-1} (\epsilon^{-1})_{\perp} \cdot \mathbf{D}_{\perp}$, the subscript $\perp$ on ϵ^{-1} indicating restriction to the x - y plane (i.e. in Cartesian coordinates striking out the third row and column). The transverse components of (3) and (4) are then

$$\epsilon_0^{-1} \hat{\mathbf{z}} \frac{d}{dz} \times (\epsilon^{-1})_{\perp} \cdot \mathbf{D}_{\perp} = i\omega\mu_0 \mathbf{H}_{\perp}, \quad (5)$$

$$\hat{\mathbf{z}} \frac{d}{dz} \times \mathbf{H}_{\perp} = -i\omega \mathbf{D}_{\perp} \quad (6)$$

Defining $\mathbf{E}_{\perp} = \epsilon_0^{-1} (\epsilon^{-1})_{\perp} \cdot \mathbf{D}_{\perp}$, $\mathbf{H}'_{\perp} = (\mu_0/\epsilon_0)^{1/2} \mathbf{H}_{\perp}$ and $k_0 = \omega (\epsilon_0\mu_0)^{1/2}$, we then have

$$\hat{\mathbf{z}} \frac{d}{dz} \times \mathbf{E}_{\perp} = ik_0 \mathbf{H}'_{\perp}, \quad (7)$$

$$\hat{\mathbf{z}} \frac{d}{dz} \times \mathbf{H}'_{\perp} = -ik_0 [(\epsilon^{-1})_{\perp}]^{-1} \cdot \mathbf{E}_{\perp}. \quad (8)$$

Combining (7) and (8) yields the Helmholtz equation for the transverse electric field components as

$$\frac{d^2 \mathbf{E}_{\perp}}{dz^2} + k_0^2 [(\epsilon^{-1})_{\perp}]^{-1} \cdot \mathbf{E}_{\perp} = 0. \quad (9)$$

In Cartesian coordinates

$$[(\epsilon^{-1})_{\perp}]^{-1} = \mathbf{R}_{\perp} \cdot \begin{pmatrix} \tilde{\epsilon} & 0 \\ 0 & \epsilon_2 \end{pmatrix} \cdot \mathbf{R}_{\perp}^{-1}, \quad (10)$$

where

$$\tilde{\epsilon} = \frac{\epsilon_a \epsilon_c}{\epsilon_c \cos^2 \chi + \epsilon_a \sin^2 \chi}. \quad (11)$$

The Helmholtz equation for light propagating axially through a SCM has thus been derived.

3 The Coupled Wave Theory of Structurally Chiral Media

The application of coupled wave theory (CWT) as a means of approximating the solution of Eq. (9) was introduced in [14, 16, 18]. Here we show how CWT reduces the problem of axial propagation through an absorptive SCM of thickness L to that of a complex scalar grating.

Expanding $\mathbf{E}_{\perp}$ in terms of forward and backward propagating modes

$$\mathbf{E}_{\perp} = \mathbf{A}^+ e^{ikz} + \mathbf{A}^- e^{-ikz}, \quad (12)$$

where $k = \bar{n}k_0$. The average refractive index seen by the axially propagating light is $\bar{n} = [(\tilde{\epsilon} + \epsilon_b)/2]^{1/2} \approx (\tilde{n} + n_b)/2$ for small birefringence, $\delta n = \tilde{n} - n_b$, where $\tilde{n} = \tilde{\epsilon}^{1/2}$ and $n_b = \epsilon_b^{1/2}$.

When loss (or gain) is included, the imaginary parts of the principal dielectric constants are non-zero, and both $\bar{n}$ and δn are complex. The most useful basis on which to express $\mathbf{A}^{\pm}$ is a circular basis such that

$$\mathbf{A}^+ = \frac{A_L^+}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{A_R^+}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad \mathbf{A}^- = \frac{A_L^-}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{A_R^-}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}. \quad (13)$$

If $A_L^+ = 1$ and $A_R^+ = 0$ then the real part of $\mathbf{A}^+ e^{ikz} = [\cos kz, -\sin kz]^T / \sqrt{2}$ describes a *left-handed spatial helix*. The rotation $\mathbf{R}_\perp$ is usefully expanded as

$$\mathbf{R}_\perp(z) = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} e^{ipz} + \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} e^{-ipz}. \quad (14)$$

After inserting (12)–(14) into (9) and equating phase matched terms we find that $dA_L^\pm/dz = 0$ and

$$\frac{d}{dz} \begin{bmatrix} A_R^+ \\ A_R^- \end{bmatrix} = \begin{bmatrix} 0 & i\kappa e^{-i\delta kz} \\ -i\kappa e^{i\delta kz} & 0 \end{bmatrix} \begin{bmatrix} A_R^+ \\ A_R^- \end{bmatrix} e^{i\delta kz}, \quad (15)$$

where the coupling constant $\kappa \approx \pi \delta n / \lambda_0$, and the detuning $\delta k = 2(k - p)$, λ_0 being the free-space wavelength. In the presence of loss or gain, κ is complex. Eq. (15) thus shows that for light that is co-handed with the medium chirality, the problem has been reduced to a scalar grating. The solution of (15), together with $E_L^\pm(z) = E_L^\pm(0) \exp(\pm ikz)$, enables the field components to be propagated from $z = 0$ to $z = L$ according to

$$\begin{bmatrix} E_L^+ \\ E_L^- \\ E_R^+ \\ E_R^- \end{bmatrix}_{z=d^-} = \begin{bmatrix} e^{ikL} & 0 & 0 & 0 \\ 0 & e^{-ikL} & 0 & 0 \\ 0 & 0 & \mathcal{P}^+ & \mathcal{Q}^+ \\ 0 & 0 & \mathcal{Q}^- & \mathcal{P}^- \end{bmatrix} \begin{bmatrix} E_L^+ \\ E_R^+ \\ E_L^- \\ E_R^- \end{bmatrix}_{z=0^+}. \quad (16)$$

The notation $z = 0^+$ and $z = L^-$ indicates the location of fields at the boundaries of the SCM, just inside the medium. Also,

$$\mathcal{P}^\pm = e^{\pm ipL} \left[\cosh(\Delta L) \pm i \frac{\delta k}{2\Delta} \sinh(\Delta L) \right], \quad (17)$$

$$\mathcal{Q}^\pm = \pm \frac{ie^{\pm ipL} \kappa}{\Delta} \sinh(\Delta L), \quad (18)$$

and

$$\Delta = +[\kappa^2 - (\delta k/2)^2]^{1/2}. \quad (19)$$

From Eq. (16) we see that within the SCM, left circularly polarized light propagates as if in a homogeneous, isotropic medium with refractive index $\bar{n}$. Forward and reverse propagating right circularly polarized modes are coupled to each other exactly as for a scalar Bragg grating [13], responding to the periodic dielectric perturbation of amplitude $\delta n/2$. Solving Eq. (16) for the chiral reflection and transmission coefficients yields

$$r_c = \frac{E_R^-(0)}{E_R^+(0)} = -\frac{\mathcal{Q}^-}{\mathcal{P}^-} = \frac{i\kappa \sinh(\Delta L)}{\Delta \cosh(\Delta L) - i(\delta k/2) \sinh(\Delta L)}, \quad (20)$$

and

$$t_c = \frac{E_R^+(L)}{E_R^+(0)} = \frac{1}{\mathcal{P}^-} = e^{ipL} [\cosh(\Delta L) + i(\delta k/2\Delta) \sinh(\Delta L)]^{-1}, \quad (21)$$

where we have used the fact that $(\mathcal{P}^+ \mathcal{P}^- - \mathcal{Q}^+ \mathcal{Q}^-) = 1$. The chiral grating Bragg resonance wavelength λ_0^{Br} is found by setting $\delta k = 0$ and is given by

$$\lambda_0^{Br} = \text{Re}(\bar{n}) L_p, \quad (22)$$

The bandwidth $\Delta\lambda_0$, of the Bragg resonance is found by looking at the difference between the wavelengths that satisfy $\Delta = 0$ in Eq. (19) and is given by

$$\Delta\lambda_0 = 2\text{Re}(\delta n) L_p. \quad (23)$$

On resonance right circularly polarized light incident to a right-handed SCF undergoes a distributed Bragg reflection within the SCF as for a scalar Bragg grating [13], whereas incident left circularly polarized light propagates through the SCF as for an isotropic medium of refractive index $\bar{n}$. If the SCM is surrounded by a medium that is not index matched to the SCM then field matching at the interfaces must be taken into account as discussed next.

4 Derivation of Remittances from a SCM

The apparently simple task of accounting for reflections from interfaces between the SCM and surrounding isotropic media for axial propagation is complicated by the fact these reflections are chirality reversing, whereas distributed reflections within the SCM are chirality preserving. This results in a complicated set of possible propagation paths. The aim of this section is to elucidate all these possible paths in a self-consistent way. This will enable us to achieve simple expressions for all eight reflectivities and transmissivities (e.g. incident left into reflected right, etc.) which we collectively call *remittances*. The ingredients for deriving simple expressions for the various remittances are:

1. *Localized* reflection/transmission at the boundaries of the SCM is determined by the Fresnel reflection/transmission coefficients.
2. *Distributed* reflection/transmission from/through the SCM for polarization co-handed with the medium's chirality is determined from coupled wave theory (c.f. Eqs. (20) and (21) above).
3. *Internal* reflection/transmission coefficients are found by self-consistently summing the amplitudes over all possible pathways. The phase accumulated by light that is contra-handed to the medium's chirality in traversing the medium is $\exp i\phi$, where $\phi = \bar{n}k_0L$.
4. *Resultant* reflection/transmission coefficients are found by multiplying the internal reflection/transmission coefficients by the appropriate interface transmission coefficients to account for entrance to and exit from the medium.

Accordingly, per Step 1 the localized reflection/transmission coefficients are defined as

$$r_{1c} = \frac{n_1 - \bar{n}}{n_1 + \bar{n}} = -r_{c1} , \quad (24)$$

$$r_{2c} = \frac{n_2 - \bar{n}}{n_2 + \bar{n}} = -r_{c2} , \quad (25)$$

$$t_{1c} = \frac{2n_1}{n_1 + \bar{n}} = \left(\frac{n_1}{\bar{n}}\right) t_{c1} , \quad (26)$$

$$t_{2c} = \frac{2n_2}{n_2 + \bar{n}} = \left(\frac{n_2}{\bar{n}}\right) t_{c2} . \quad (27)$$

Per Step 2, distributed reflection and transmission is then quantified by the coefficients r_c and t_c given in Eqs. (20) and (21) respectively.

To illustrate Step 3 consider the pathways identified in Fig. 2 for the internal reflection coefficients r_{LL}^{int} and r_{LR}^{int} . From the diagram, simultaneous equations for these reflection coefficients are constructed as

$$\mathbf{v}_{rL} = \begin{bmatrix} r_{LL}^{int} \\ r_{LR}^{int} \end{bmatrix} = \begin{bmatrix} e^{i\phi} r_{c2} t_{c1} & r_c r_{c2}^2 e^{2i\phi} r_{c1} \\ r_c r_{c1} & t_c r_{c2} e^{i\phi} r_{c1} \end{bmatrix} \begin{bmatrix} r_{LL}^{int} \\ r_{LR}^{int} \end{bmatrix} + \begin{bmatrix} r_c r_{c2}^2 e^{2i\phi} \\ t_c r_{c2} e^{i\phi} \end{bmatrix} . \quad (28)$$

The solution of Eq. (28) for the vector $\mathbf{v}_{rL}$ can be expressed as

$$\mathbf{v}_{rL} = \mathbf{M} \cdot \mathbf{c}_{rL} , \quad (29)$$

where

$$\mathbf{M} = D^{-1} \begin{bmatrix} 1 - t_c e^{i\phi} r_{c2} r_{c1} & r_c r_{c2}^2 e^{2i\phi} r_{c1} \\ r_c r_{c1} & 1 - t_c e^{i\phi} r_{c2} r_{c1} \end{bmatrix} , \quad \mathbf{c}_{rL} = \begin{bmatrix} r_c r_{c2}^2 e^{2i\phi} \\ t_c r_{c2} e^{i\phi} \end{bmatrix} , \quad (30)$$

and

$$D = \left(1 - t_c r_{c1} e^{i\phi} r_{c2}\right)^2 - \left(r_c r_{c1} e^{i\phi} r_{c2}\right)^2 . \quad (31)$$

The solutions for the remaining remittance vectors,

$$\mathbf{v}_{rR} = \begin{bmatrix} r_{RL}^{int} \\ r_{RR}^{int} \end{bmatrix} , \quad \mathbf{v}_{tL} = \begin{bmatrix} t_{LL}^{int} \\ t_{LR}^{int} \end{bmatrix} , \quad \mathbf{v}_{tR} = \begin{bmatrix} t_{RL}^{int} \\ t_{RR}^{int} \end{bmatrix} , \quad (32)$$

differ from Eq. (29) only in the form of the vector on the right hand side (i.e. the matrix $\mathbf{M}$ is the same), as may be adduced from Figs. 3-5. From the figures these vectors are

$$\mathbf{c}_{rR} = \begin{bmatrix} e^{i\phi} r_{c2} t_c \\ r_c \end{bmatrix}, \quad \mathbf{c}_{tL} = \begin{bmatrix} e^{i\phi} \\ 0 \end{bmatrix}, \quad \mathbf{c}_{tR} = \begin{bmatrix} e^{i\phi} r_{c2} r_c \\ t_c \end{bmatrix}. \quad (33)$$

Now applying Step 4 to the remittance vectors $\mathbf{v}_{rL}$, $\mathbf{v}_{rR}$, $\mathbf{v}_{tL}$ and $\mathbf{v}_{tR}$, the final formulae for the resultant reflection and transmission coefficients are straightforwardly obtained. For example

$$r_{LL} = t_{1c} r_{LL}^{int} t_{c1} = D^{-1} t_{1c} r_c r_{c2}^2 e^{2i\phi} t_{c1}, \quad (34)$$

the other remittances then being obtained similarly as

$$r_{LR} = r_{1c} + D^{-1} t_{1c} \left[t_c r_{c2} e^{i\phi} + r_{c1} r_{c2}^2 e^{2i\phi} (r_c^2 - t_c^2) \right] t_{c1} = r_{RL}, \quad (35)$$

$$r_{RR} = D^{-1} t_{1c} r_c t_{c1}, \quad (36)$$

$$t_{LL} = D^{-1} t_{1c} \left(1 - t_c e^{i\phi} r_{c2} r_{c1} \right) e^{i\phi} t_{c2}, \quad (37)$$

$$t_{LR} = D^{-1} t_{1c} r_c r_{c1} e^{i\phi} t_{c2} = \left(\frac{r_{c1}}{r_{c2}} \right) t_{RL}, \quad (38)$$

$$t_{RR} = D^{-1} t_{1c} \left[t_c + r_{c1} e^{i\phi} r_{c2} (r_c^2 - t_c^2) \right] t_{c2}. \quad (39)$$

Eqs. (34)-(39) are the key results of this paper. The leading r_{1c} term on the right of the expression for $r_{LR} = r_{RL}$ accounts for the direct reflection at the first interface. Note that $r_{RL} = r_{LR}$, but $t_{RL} = t_{LR}$ only if the indices of the surrounding media are equal, i.e. $n_1 = n_2$. Notwithstanding their more general applicability, Eqs. (34)-(39) are significantly simpler than the corresponding formulae in [18] (Eqs. (20)-(26)). Each remittance is here expressed compactly in terms of the constituent reflection/transmission coefficients of the chiral medium and the interfaces. Moreover, Eqs. (34)-(39) apply when $n_1 \neq n_2$, and for SCM with isotropic absorption or gain.

5 Exact Theory

The analytic solution of Eqs. (3) and (4) of axial propagation through SCMs has been reproduced extensively in the literature (see [12] and references therein) since the first detailed exposition in [21]. However, we give a compact summary here for completeness. After reducing the problem to the transverse projection of Eqs. (7) and (8), the auxiliary fields $\mathbf{e}$ and $\mathbf{h}$ are introduced as

$$\mathbf{e} = \mathbf{R}_{\perp}^{-1}(z) \cdot \mathbf{E}_{\perp}, \quad (40)$$

$$\mathbf{h} = \mathbf{R}_{\perp}^{-1}(z) \cdot \mathbf{H}'_{\perp}, \quad (41)$$

reduces Eqs. (7) and (8) to the system

$$\frac{d}{dz} \begin{bmatrix} e_x \\ e_y \\ h_x \\ h_y \end{bmatrix} = i \begin{bmatrix} 0 & -ip & 0 & k_0 \\ ip & 0 & -k_0 & 0 \\ 0 & -k_0 \epsilon_b & 0 & -ip \\ k_0 \tilde{\epsilon} & 0 & ip & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ h_x \\ h_y \end{bmatrix}, \quad (42)$$

where importantly the matrix on the right (denoted $\mathbf{G}$) is independent of z . The original transverse field components are therefore given by

$$\begin{bmatrix} \mathbf{E}_{\perp} \\ \mathbf{H}'_{\perp} \end{bmatrix}_{z=L} = \begin{bmatrix} \mathbf{R}_{\perp}(L) & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\perp}(L) \end{bmatrix} e^{i\mathbf{G}L} \begin{bmatrix} \mathbf{E}_{\perp} \\ \mathbf{H}'_{\perp} \end{bmatrix}_{z=0}, \quad (43)$$

where $\mathbf{0}$ is the 2×2 null matrix, and $\exp(i\mathbf{G}L)$ is a well-defined analytic function of the matrix $\mathbf{G}$ [22]. If the SCM is surrounded by isotropic media of refractive indices $n_{1,2}$, and if $a_{x,y}, r_{x,y}, t_{x,y}$ represent the amplitudes of the Cartesian components of respectively the incident, reflected and

transmitted electric fields, then field matching at $z = 0$ and $z = L$ requires that

$$\begin{bmatrix} \mathbf{E}_\perp \\ \mathbf{H}_\perp \end{bmatrix}_{z=0} = \begin{bmatrix} a_x + r_x \\ a_y + r_y \\ -n_1(a_y - r_y) \\ n_1(a_x - r_x) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{E}_\perp \\ \mathbf{H}_\perp \end{bmatrix}_{z=L} = \begin{bmatrix} t_x \\ t_y \\ -n_2 t_y \\ n_2 t_x \end{bmatrix}. \quad (44)$$

For known incident amplitudes $a_{x,y}$, inserting (44) into (43) provides four simultaneous equations for the unknowns $r_{x,y}, t_{x,y}$. The linear reflection and transmission coefficients are then defined according to

$$\begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}. \quad (45)$$

For example, r_{xy} is the amplitude reflection into x polarization for incident y polarization. Reflection and transmission coefficients defined as above on a Cartesian basis may be transformed to a circular basis by noting that

$$\begin{bmatrix} r_x \\ r_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} r_L \\ r_R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_x, t_x \\ a_y, t_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} a_L, t_L \\ a_R, t_R \end{bmatrix}. \quad (46)$$

Hence

$$\begin{bmatrix} r_{LL} & r_{LR} \\ r_{RL} & r_{RR} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \quad (47)$$

and

$$\begin{bmatrix} t_{LL} & t_{LR} \\ t_{RL} & t_{RR} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad (48)$$

Although the above procedure ultimately yields analytic formulae for the circular reflection and transmission coefficients, the resulting expressions are too unwieldy for practical design. The theory is, however, exact and valid for arbitrary birefringence, and therefore provides a useful benchmark against which other theoretical approaches can be tested.

6 Comparison of Methods

Here we compare the reflection/transmission coefficients obtained using the approximate formulae of Eqs. (39)-(34) with those obtained using the method of Section 5. We take the typical parameters of Table 1. Small isotropic absorption is included in the principal dielectric constants of the SCM. For these parameters we have, according to Eqs. (22) and (23) that the Bragg resonance occurs at $\lambda_0^{Br} = 519$ nm and is of bandwidth $\Delta\lambda_0^{Br} = 16$ nm.

Structural period of chiral medium	L_z	300nm
Principal dielectric constants of chiral medium	ϵ_a	$3.2 + 0.02i$
	ϵ_b	$2.9 + 0.02i$
	ϵ_c	$2.8 + 0.02i$
Rise angle	χ	30°
Depth of chiral medium	L	6 μm
Refractive index of incident medium	n_1	1
Refractive index of transmission medium	n_2	2

Table 1: Data used for calculations

The reflected and transmitted powers are given by

$$\begin{bmatrix} R_{LL} & R_{LR} \\ R_{RL} & R_{RR} \end{bmatrix} = \begin{bmatrix} |r_{LL}|^2 & |r_{LR}|^2 \\ |r_{RL}|^2 & |r_{RR}|^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{LL} & T_{LR} \\ T_{RL} & T_{RR} \end{bmatrix} = \left(\frac{n_2}{n_1}\right) \begin{bmatrix} |t_{LL}|^2 & |t_{LR}|^2 \\ |t_{RL}|^2 & |t_{RR}|^2 \end{bmatrix}. \quad (49)$$

The calculations using Eqs. (34)-(39) in Eq. (49) are shown in Figs. 6 and 7. The corresponding calculations using the method of Section 5 are identical within the resolution of the plots, the differences being detailed in Figs. 8 and 9 respectively. The maximal discrepancy

of about 0.6% is evident in the co-handed remittances R_{RR} . The typical discrepancy is about 0.1%. In fact, the discrepancy decreases with increasing absorption. With zero absorption the maximal discrepancy is higher but still less than 1%. As the birefringence increases, it would be anticipated that the discrepancy would increase further since then the inherent assumption in coupled wave theory, that $|\delta n| \ll \bar{n}$ starts to break down. Hodgkinson et al [25] have achieved a relative birefringence of $|\delta n|/\bar{n} = 0.072$ and we have found that the formulae still perform with about 5% maximal discrepancy in this case, the principal discrepancy being at the band edges, as is already apparent from Figs. 8 and 9. Away from the band edges the discrepancy is much smaller. In fact, numerical difficulties associated with the calculation of the matrix exponential in Eq. (43) also start to arise in attempting exact calculations at high relative birefringence as was discussed in [18].

7 Special Cases, Energy Conservation and Design Strategies

Eqs. (34)-(39) reassuringly reproduce several known special cases:

1. Turning off the chiral medium by setting $\delta n = 0$ determines that $\kappa = 0$ and $\Delta = i(k - p)$, and in turn that $\mathcal{Q}^\pm = 0$, $\mathcal{P}^\pm = e^{i\phi}$, $r_c = 0$ and $t_c = e^{i\phi}$. The matrix in (16) therefore reduces to the field transfer matrix across an isotropic medium of index $\bar{n}$. Moreover, substituting these simplifications into Eqs. (31) and (34)-(39) produces

$$D = \left(1 - r_{c1}e^{2i\phi}r_{c2}\right)^2, \quad (50)$$

and

$$r_{LL} = r_{RR} = t_{LR} = 0 , \quad (51)$$

$$r_{LR} = r_{1c} + \frac{r_{c2}t_{1c}t_{c1}e^{2i\phi}}{1 - r_{c1}r_{c2}e^{2i\phi}} = r_{RL} , \quad (52)$$

$$t_{LL} = \frac{t_{1c}t_{c2}e^{i\phi}}{1 - r_{c1}r_{c2}e^{2i\phi}} , \quad (53)$$

$$t_{RR} = \frac{t_{1c}t_{c2}e^{i\phi}}{1 - r_{c1}r_{c2}e^{2i\phi}} , \quad (54)$$

which are the expected results for an isotropic thin film.

2. Index matching the SCM to the surrounding media ($\bar{n} = n_1 = n_2$) reduces Eqs. (34)-(39) to $r_{LL} = r_{LR} = t_{LR} = 0$, $r_{RR} = r_c$, $t_{LL} = e^{i\phi}$ and $t_{RR} = t_c$, i.e. right circular polarization is reflected and transmitted precisely as prescribed by the SCM, whereas left circular polarization is transmitted as if no interfaces were present and the light is simply propagating through a single medium of index $\bar{n}$.
3. Letting the SCM thickness $L \rightarrow \infty$ with a small amount of absorption present sets $e^{i\phi} \rightarrow 0$ and $\mathcal{P}^- \rightarrow \infty$. Hence $t_c \rightarrow 0$, $r_c \rightarrow i\kappa/(\Delta - i\delta k/2)$ and Eqs. (34)-(39) then become

$$r_{LR} = r_{1c} = r_{RL} , \quad (55)$$

$$r_{RR} = \frac{i\kappa t_{1c}t_{c1}}{\Delta - i\delta k/2} , \quad (56)$$

with the other remittances vanishing. These were the results obtained for a chiral medium half space in [19].

A consistency check on the remittance formulae is that they must satisfy energy conservation, i.e.

$$|r_{LL}|^2 + |r_{RL}|^2 + \left(\frac{n_2}{n_1}\right) (|t_{LL}|^2 + |t_{RL}|^2) = 1 , \quad (57)$$

$$|r_{LR}|^2 + |r_{RR}|^2 + \left(\frac{n_2}{n_1}\right) (|t_{LR}|^2 + |t_{RR}|^2) = 1 . \quad (58)$$

However, a third energy conservation rule has recently been discovered [10] that has apparently been overlooked in the previous SCM literature, namely:

$$r_{LL}r_{LR}^* + r_{RL}r_{RR}^* + \left(\frac{n_2}{n_1}\right)(t_{LL}t_{LR}^* + t_{RL}t_{RR}^*) = 0. \quad (59)$$

This relation represents the redistribution of energy between the L and R outputs due to interference effects, and this equation simply states that the sum of these energy transfers is zero. We have verified that the conservation relations are algebraically satisfied when Eqs. (34)-(39) are substituted in to Eqs. (57), (58) and (59).

In designing a structurally chiral thin film for particular applications, a number of observations arise from direct examination of Eqs. (34)-(39):

1. Imbalancing the indices of the surrounding media induces asymmetries between the contrahanded transmissions, i.e. $t_{LR} \neq t_{RL}$.
2. r_{RR} can only be switched off by switching off the chiral medium (i.e. by setting $\delta n = 0$).
3. t_{LR} (resp. t_{RL}) can be switched off by setting $n_1 = \bar{n}$ ($n_2 = \bar{n}$). Both t_{LR} and t_{RL} can be switched off by setting $\delta n = 0$.
4. When n_1 and n_2 are distinct from n_c , then according to Eq. (35) $r_{LR} = r_{RL} = 0$ can still be achieved, although the condition is then wavelength dependent.
5. The quantity D in Eq. (31) is a key parameter. Setting $D = 0$ in the presence of gain is the oscillation condition for a distributed chiral laser.
6. From Eq. (39) we see that $t_{LL} = 0$ only in the presence of gain.

These observations cannot easily be deduced from the exact theory of Section 5.

8 Conclusion

Physics always aims at simplification, and in the theory of structurally chiral dielectric media we can now claim a significant milestone. The formulae developed here reproduce, in almost every detail, the results presented in the first detailed expositions of SCM theory [21]. Though not examined here, it is a relatively straightforward task to modify the method applied here to structurally chiral dielectric media optimized for elliptical polarization [23] by casting the coupled wave theory on an elliptical basis, rather than the circular basis used here. As with all coupled wave techniques, the theory is only valid for $\delta n \ll \bar{n}$, although this was found previously to embrace the full range of birefringence encountered in practically manufactured SCM's [18]. Moreover, by expanding Eqs. (34)-(39) in various orders of δn and $n_{1,2} - \bar{n}$ all the special cases of the previous literature [14, 16, 18, 19] are reproduced.

Finally we note that the theory also applies to gain media, the design of a chiral laser [24] being directed by the oscillation condition $D = 0$ in Eq. (31). This will be pursued in further work.

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References

- [1] N. O. Young and J. Kowal, ‘Optically active fluorite films’, *Nature*, **183**, 104-105, (1959).

- [2] K. Robbie, M. J. Brett, and A. Lakhtakia, ‘Chiral sculptured thin films’, *Nature (London)*, **384**, 616, (1996).
- [3] P. Yeh and C. Gu, *Optics of liquid crystal displays*, Wiley, New York, NY, USA, (1999).
- [4] Y.J. Park, K.M.A. Sobahan and C.K.Kwon Hwangbo, ‘Wideband circular polarization reflector fabricated by glancing angle deposition’, *Optics Express*, **16**, 5186–5192, (2008).
- [5] Lakhtakia,A., McCall,M.W., Sherwin,J.A., Wu,Q.H., and Hodgkinson,I.J., ‘Sculptured-thin-film Spectral Holes for Optical Sensing of Fluids’, *Optics Communications*, **194**, 33-46, (2001).
- [6] Hodgkinson,I.J., Wu,Q-H, Lakhtakia,A. and McCall,M., ‘Spectral Hole Filter Fabricated Using Sulptured Thin Film Technology’, *Optics Communications*, **169**, 79–84, (2000).
- [7] Hodgkinson,I.J., Wu,Qi-H., De Silva,L., Arnold,M., McCall,M.W., Lakhtakia,A., ‘Supermodes of Chiral Photonic Filters with Combined Twist and Layer Defects’, *Physical Review Letters*, **91**, 223903-1,4, (2003).
- [8] Lakhtakia,A., Venugopal,V.C. and McCall,M.W., ‘Spectral Holes In Bragg Reflection From Chiral Sculptured Thin Films: Circular Polarization Filters’, *Optics Communications*, **177**, 57-68, (2000).
- [9] V. I. Kopp, V. M. Churikov, G. Zhang, J. Singer, C. W. Draper, N. Chao, D. Neugroschl, and A. Z. Genack, ‘Chiral fiber gratings: perspectives and challenges for sensing applications’ (Invited Paper), *Proceedings of SPIE*, 6619, 66190B (2007).
- [10] McCall, M.W. and Hodgkinson, I. ‘Polarized Light in Transmission and Reflection’, *Submitted to European Journal of Physics, special feature on Nanotechnology*, (2008).

- [11] Sam Lowrey, Lakshman De Silva, Ian Hodgkinson and John Leader, ‘Observation and modeling of polarized light from scarab beetles’, *J. Opt. Soc. Am. A*, **24**, 2418–2425, (2007).
- [12] A. Lakhtakia and R. Messier, ‘Sculptured Thin Films: Nanoengineered Morphology and Optics’, SPIE Press, (2005).
- [13] McCall, M.W., ‘On The Application Of Coupled Mode Theory For The Modelling Of Fibre-Bragg Gratings’, *Journal of Lightwave Technology*, **18**, 236–242, (2000).
- [14] M.W. McCall and A. Lakhtakia, ‘Development and assessment of coupled wave theory of axial propagation in thin-film helicoidal bianisotropic mediums. Part 1: reflectances and transmittances’, *Journal of Modern Optics* **47**, 973–991, (2000). *Erratum*: **50**, 2807, (2003).
- [15] McCall, M. W., ‘Axial Electromagnetic Wave Propagation in Inhomogeneous Dielectrics’, *Mathematical and Computer Modelling*, **34**, 1483–1497, (2001).
- [16] M.W. McCall and A. Lakhtakia, ‘Development and assessment of coupled wave theory of axial propagation in thin-film helicoidal bi-anisotropic media. Part 2: Dichroisms, ellipticity transformation and optical rotation’, *Journal of Modern Optics* **48** (2001), 143–158.
- [17] Hodgkinson, I. J., Wu, Qi-H., Arnold, M., McCall, M. W. and Lakhtakia, A., ‘Chiral Mirror and Optical Resonator Designs for Circularly-Polarized Light: Suppression of Cross-Polarized Reflectances and Transmittances’, *Optics Communications*, **210**, 201–211, (2002).
- [18] M.W. McCall and A. Lakhtakia, ‘Explicit Expressions for Spectral Remittances of Axially Excited Chiral Sculptured Thin Films’, *J Modern Opt* **51** (2004), 111–127.

- [19] McCall, M. W. and Lakhtakia, A., ‘Analysis of Plane Wave Light Normally Incident to an Axially Excited Structurally Chiral Half-Space’, *Journal of Modern Optics*, **52**, 541-550, (2005).
- [20] Lakhtakia, A. and McCall, M. W., ‘Simple Expressions for Bragg reflection from axially Excited chiral sculptured thin films’, *Journal of Modern Optics*, **49**, 1525-1535, (2002).
- [21] Lakhtakia, A. and Weiglhofer, W. S., ‘On light propagation in helicoidal bianisotropic mediums’, *Proc. R. Soc. Lond.*, **A448**, 419–437, (1995).
- [22] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer, New York, NY, USA, 1983, Sec. 1.1.
- [23] Hodgkinson, I. J., De Silva, L., Lakhtakia, A. and McCall, M. W., ‘Structurally-perturbed chiral Bragg reflectors for elliptically polarized light’, *Optics Letters*, **30**, 2629-2631, (2005).
- [24] Chanishvili, A., Chilaya, G., Petriashvili, G., Barberi, R., Bartolino, R., Cipparrone, G. and Mazzulla, A., ‘Laser emission from a dye-doped cholesteric liquid crystal pumped by another cholesteric liquid crystal laser’, *Appl. Phys. Lett.*, **85**, 3378–3380, (2004).
- [25] Hodgkinson, I., Wu, Q. H., Knight, B., Lakhtakia, A., and Robbie, K., ‘Vacuum deposition of chiral sculptured thin films with high optical activity’, *Appl. Optics*, **39**, 642–649, (2000).

Figure Captions

Figure 1: Geometry of Chiral Sculptured Thin Film as described by Eqs. (1) and (2). For ease of depiction the case of zero rise angle ($\chi = 0$) is shown.

Figure 2: Propagation pathways linking the reflection coefficients r_{LL}^{int} and r_{LR}^{int} . Diagrams (a) and (b) are used to derive Eq. (28).

Figure 3: Propagation pathways linking the reflection coefficients r_{RL}^{int} and r_{RR}^{int} . Diagrams (a) and (b) are used to derive the equation $\mathbf{v}_{rR} = \mathbf{M} \cdot \mathbf{c}_{rR}$ (c.f Eqs. (28)-(33)).

Figure 4: Propagation pathways linking the transmission coefficients t_{LL}^{int} and t_{LR}^{int} . Diagrams (a) and (b) are used to derive the equation $\mathbf{v}_{tL} = \mathbf{M} \cdot \mathbf{c}_{tL}$ (c.f Eqs. (28)-(33)).

Figure 5: Propagation pathways linking the transmission coefficients t_{RL}^{int} and t_{RR}^{int} . Diagrams (a) and (b) are used to derive the equation $\mathbf{v}_{tR} = \mathbf{M} \cdot \mathbf{c}_{tR}$ (c.f Eqs. (28)-(33)).

Figure 6: Reflections calculated by inserting the reflection coefficients (34)-(36) in Eq. (49)a. Data is given in Table 1.

Figure 7: Transmissions calculated by inserting the transmission coefficients (37)-(39) in Eq. (49)b. Data is given in Table 1.

Figure 8: The computed differences between the Reflections of Fig. 6 and those calculated using the exact method of Section 5.

Figure 9: The computed differences between the Transmissions of Fig. 7 and those calculated using the exact method of Section 5.

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