Basket CDS Pricing with Interacting Intensities

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Abstract. In this paper we propose a factor contagion model for correlated defaults. The model covers the heterogeneous conditionally independent portfolio and the factor infectious default portfolio as special cases. The model assumes that the hazard rate processes are driven by external common factors as well as defaults of other names in the portfolio. The total hazard construction method is used to derive the joint distribution of default times. The basket CDS rates can be computed analytically for homogeneous contagion portfolios and recursively for general factor contagion portfolios. We extend the results to include the interacting counterparty risk and the stochastic intensity process. We compare the analytic results with the Monte Carlo results and discuss the numerical stability problem for large portfolios.

Keywords. factor contagion model, basket CDS, analytic pricing formula, counterparty risk, stochastic intensity.

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1 Introduction

Portfolio credit derivatives have been popular in financial industry for managing and hedging portfolio credit and default risk. The recent credit crunch of the sub-prime mortgage debt crisis is essentially due to overestimation of asset value and underestimation of correlated default risk. This highlights the importance of thorough characterization of dependence risk in asset valuation. The sources of dependence come from the common macro economic factors that affect all firms and from the direct interaction of firms such as supply chain links and counterparty exposures. There are mainly three

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approaches to model the default correlation in the literature: copula, conditional independence, and contagion.

The copula model constructs the joint distribution of default times by combining marginal distributions of default times of individual names with a copula which completely determines the dependence structure. A well known copula is the normal copula which is used to model the default correlation of CreditMetrics, see Li (1999). Other copulas, such as Archimedean copulas, are often used in stress testing and the extreme value theory.

The conditional independence model is the most popular one in correlation modelling. It assumes that some systematic factors affect default intensities of all names in the portfolio. Conditional on the realization of the common factor, the default intensities and default times of names are independent of each other. The joint distribution of default times can be expressed semi-analytically, and analytically in the limiting case for homogeneous portfolios, see Gregory (2003) and Schonbucher (2004). It is often difficult to compute the joint distribution of default times for large heterogeneous portfolios due to combinatorial number of summation elements. Some efficient approximation techniques have been developed to alleviate the computational burden, such as the tail approximation method (Glasserman (2004)) and the hybrid normal approximation method (Zheng (2006)). One shortcoming of the conditional independence model is that it cannot deal with the direct interaction of obligors, which may underestimate the portfolio risk and economic capital, see Das et al. (2007) for empirical studies and other references in BCBS (2006).

The contagion model studies the direct interaction of names in which the default probability of one name may change upon defaults of some other names in the portfolio, and vice versa, and “infectious defaults” may develop, see Davis and Lo (2001). The “looping” dependence of default times of hazard rate processes makes difficult in characterizing the joint distribution of default times. Jarrow and Yu (2001) suggest the primary secondary framework for the interaction of default intensities, which excludes cyclical default dependence, and derive the joint distribution. Frey and Backhaus (2004) apply the Markov process technique and the Kolmogorov equations to analyze in detail a model where the interaction between firms is of the mean-field type and use the Monte Carlo method for pricing portfolio credit products. Herbertsson and Rootzen (2006) also apply the Markov chain technique and the matrix exponentials to give the analytic pricing formula for basket CDS rates. Leung and Kwok (2005) apply the CGH formula (Collin-Dufresne, et al. (2002)) to derive the joint density function of three names in the contagion model and use it to price single-name CDSs. Yu

(2007) applies the total hazard construction method (Shaked and Shanthikumar (1987)) to get the joint density function of three names and uses it to price bonds and single-name CDSs. Yu (2007) also suggests the Monte Carlo method for pricing basket CDSs.

In this paper we study the joint distribution of default times for factor contagion models which unifies the factor model and the contagion model in the following sense: the portfolio is made up of subportfolios which are conditionally independent and obligors of each subportfolio are correlated in a contagion way. If each subportfolio contains only one obligor, then we recover the heterogeneous conditionally independent model; if there is only one subportfolio then we recover the contagion model. This approach provides a realistic characterization of dependence structure of obligors. For example, iTraxx Europe is made up of firms from different sectors, all firms are subject to risk of macro economic factors, firms within a sector are correlated to each other through business link or competition but are independent of firms in other sectors. To model iTraxx Europe with a homogeneous conditionally independent portfolio seems over-simplistic and may underestimate the contagion risk.

To strike the balance between the generality of model and the ease of computation and calibration, we assume that obligors in the same subportfolio have the same default intensity and the same contagion rate. The benefit of the homogeneous contagion subportfolio assumption is that we can have a simple analytic formula for joint distribution of default times of obligors. This has huge computational advantage over some other methods such as the Kolmogorov equations or the matrix exponentials which are computationally expensive when the portfolio is large. The analytic formula also makes easy in finding sensitivities of the joint distribution with respect to the underlying parameters and in calibrating the model.

The paper is organized as follows: section 2 applies the total hazard construction method to find the joint distribution of default times, section 3 derives the ordered default time distribution for homogeneous contagion portfolios and suggests a recursive algorithm for general portfolios, section 4 extends the results to the stochastic intensity model and the interacting counterparty default risk model, section 5 compares numerically the analytic results with the Monte Carlo results and discusses the stability issue, section 6 gives a summary, and section 7 contains the proofs of the main results.

2 Total Hazard Construction and Default Time Distribution

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space, where P is the martingale measure and $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration satisfying the usual conditions. Let τ_i be the default time of name i , $N_i(t) = 1_{\{\tau_i \leq t\}}$ the default indicator process of name i , $\mathcal{F}_t^i = \sigma(N_i(s) : s \leq t)$ the filtration generated by default process N_i , $i = 1, \dots, n$, $X(t)$ a stochastic state process, $\mathcal{F}_t^X = \sigma(X(s) : s \leq t)$ the filtration generated by X , and $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \dots \vee \mathcal{F}_t^n$ the smallest σ -algebra needed to support $\tau_1, \dots, \tau_n$ and X . Assume that τ_i possesses a nonnegative $\mathcal{F}_t$ predictable intensity process $\lambda_i(t)$ satisfying $E(\int_0^t \lambda(s) ds) < \infty$ for all t , and the compensated process

$$M_i(t) = N_i(t) - \int_0^{t \wedge \tau_i} \lambda_i(s) ds$$

is an $\mathcal{F}_t$ martingale. The intensity processes $\lambda_i(t)$ in general depend on the state process X_t and conditional on the complete sample path of X , i.e. $\mathcal{F}_\infty^X$, the intensity processes $\lambda_i(t)$ are $\vee_i \mathcal{F}_t^i$ predictable. Any contingent claims involving default times τ_i can be priced with the law of iterated expectations. We therefore do not explicitly specify the dependence of $\lambda_i(t)$ on the state process $X(t)$ from now on.

Given $\tau_j = t_j$, $j \in J_k = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$, satisfying $0 = t_{j_0} < t_{j_1} < \dots < t_{j_k}$ and $\tau_i > t > t_{j_k}$ for $i \notin J_k$, the conditional hazard rate of τ_i at time t is given by

$$\lambda_i(t|t_{J_k}) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P(t < \tau_i \leq t + \Delta t | \tau_j = t_j, j \in J_k)$$

where t_{J_k} is a short form for $(t_{j_1}, \dots, t_{j_k})$. When $k = 0$, i.e., no defaults have occurred at time t , then $\lambda_i(t|t_{J_k})$ is the unconditional hazard rate $\lambda_i(t)$ of name i at time t . The total hazard accumulated by name i during time interval $[t_{j_k}, t_{j_k} + s]$, $s \geq 0$, is defined by

$$\Lambda_i(s|t_{J_k}) = \int_{t_{j_k}}^{t_{j_k} + s} \lambda_i(u|t_{J_k}) du.$$

The total hazard accumulated by name i by time t , given k observed defaults $\tau_j = t_j, j \in J_k$, is given by

$$\psi_i(t|t_{J_k}) = \sum_{l=0}^{k-1} \Lambda_i(t_{j_{l+1}} - t_{j_l} | t_{j_l}) + \Lambda_i(t - t_{j_k} | t_{J_k}).$$

Define the inverse functions

$$\Lambda_i^{-1}(x|t_{J_k}) = \inf\{s \geq 0 : \Lambda_i(s|t_{J_k}) \geq x\}$$

for $i \notin J_k$ and $x \geq 0$. The total hazard can be constructed as follows: Let $E_1, \dots, E_n$ be independent standard exponential variables.

Step 1. Let

$$j_1 = \operatorname{argmin}\{\Lambda_i^{-1}(E_i) : i = 1, \dots, n\}$$

and define

$$\hat{\tau}_{j_1} = \Lambda_{j_1}^{-1}(E_{j_1}) \quad (1)$$

and set $J_1 = \{j_1\}$.

Step $k+1$ ($k = 1, \dots, n-1$). Given that Steps $1, \dots, k$ have resulted in $\hat{\tau}_j, j \in J_k$. Let

$$j_{k+1} = \operatorname{argmin}\{\Lambda_i^{-1}[E_i - \psi_i(\hat{\tau}_{j_k}|\hat{\tau}_{J_k})|\hat{\tau}_{J_k}] : i \notin J_k\}$$

and define

$$\hat{\tau}_{j_{k+1}} = \hat{\tau}_{j_k} + \Lambda_{j_{k+1}}^{-1}[E_{j_{k+1}} - \psi_{j_{k+1}}(\hat{\tau}_{j_k}|\hat{\tau}_{J_k})|\hat{\tau}_{J_k}] \quad (2)$$

and set $J_{k+1} = J_k \cup \{j_{k+1}\}$.

Norros (1986), Shaked and Shanthikumar (1987), and Yu (2007) prove that $\hat{\tau} = (\hat{\tau}_1, \dots, \hat{\tau}_n)$ equals $\tau = (\tau_1, \dots, \tau_n)$ in distribution. We can therefore generate default times τ by generating $\hat{\tau}$ instead and we will not distinguish them from now on.

Consider the following intensity processes

$$\lambda_i(t) = a_{i0}(t) + \sum_{j \neq i} a_{ij}(t) 1_{\{\tau_j \leq t\}}. \quad (3)$$

If $a_{ij}(t) = 0$ for all $i \neq j$ then default times of names are independent of each other. If $a_{ij}(t) = \infty$ for all $i \neq j$ then default of one name immediately triggers defaults of all other names. The “infectious defaults” of Davis and Lo (2001) may be thought of the special case of (3).

With the total hazard method (1) and (2), we can express default times τ in terms of standard exponential variables E , and vice versa. For example, if $j_k = k$ for $k = 1, \dots, n$, i.e., τ_1 is the first default time, τ_2 the second default time, etc., then from (1) and (2) we have

$$E_k = \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_k} a_{kj}(u) du \quad \text{for } k \geq 1$$

with $\tau_0 = 0$. The Jacobi determinant of E with respect to τ is given by

$$c(\tau_1, \dots, \tau_n) = \prod_{m=1}^n \left(\sum_{l=0}^{m-1} a_{ml}(\tau_m) \right).$$

The density of τ for $\tau_1 < \dots < \tau_n$ is therefore given by $f(\tau_1, \dots, \tau_n) = c(\tau_1, \dots, \tau_n)e^{-(E_1 + \dots + E_n)}$. Substituting E_i into f we get

$$f(t_1, t_2, \dots, t_n) = c(t_1, \dots, t_n)e^{-\sum_{l=1}^n \int_0^{t_l} w_l(u) du} \quad (4)$$

for $0 < t_1 < t_2 < \dots < t_n$, where

$$w_k(u) = \left(\sum_{j=0}^{k-1} a_{kj}(u) \right) - \left(\sum_{j=k+1}^n a_{jk}(u) \right) \quad \text{for } k \geq 1.$$

The space R_+^n can be divided into $n!$ regions according to the order of $(t_1, t_2, \dots, t_n)$. The density function f in other regions can be expressed similarly with permutation. For example, if $n = 2$ and functions $a_{ij}(t)$ are constants a_{ij} , then

$$f(t_1, t_2) = \begin{cases} a_{10}(a_{20} + a_{21})e^{-(a_{10}-a_{21})t_1-(a_{20}+a_{21})t_2} & \text{if } t_1 < t_2 \\ a_{20}(a_{10} + a_{12})e^{-(a_{20}-a_{12})t_2-(a_{10}+a_{12})t_1} & \text{if } t_2 < t_1. \end{cases}$$

If $n = 3$ then in the region of $t_1 < t_2 < t_3$ we have

$$f(t_1, t_2, t_3) = a_{10}(a_{20} + a_{21})(a_{30} + a_{31} + a_{32}) \cdot e^{-[(a_{10}-a_{21}-a_{31})t_1+(a_{20}+a_{21}-a_{32})t_2+(a_{30}+a_{31}+a_{32})t_3]}.$$

If we want to find the density function in other regions, say $t_3 < t_1 < t_2$, we only need to change the indexes of the above equation from $1 \rightarrow 3$, $2 \rightarrow 1$, and $3 \rightarrow 2$, and we have

$$f(t_1, t_2, t_3) = a_{30}(a_{10} + a_{13})(a_{20} + a_{23} + a_{21}) \cdot e^{-[(a_{30}-a_{13}-a_{23})t_3+(a_{10}+a_{13}-a_{21})t_1+(a_{20}+a_{23}+a_{21})t_2]}.$$

The total hazard construction method can be used to derive the joint distribution function of default times for general intensity processes. For example, consider the following self-exciting point processes

$$\lambda_i(t) = a_{i0} + \sum_{j \neq i} a_{ij} e^{-c(t-\tau_j)} 1_{\{\tau_j \leq t\}} \quad (5)$$

where c is a positive constant representing the rate of decay. If $j_k = k$ for $k = 1, \dots, n$, then it is easy to check that

$$E_k = a_{k0}\tau_k + \sum_{j=1}^{k-1} a_{kj} \frac{1 - e^{-c(\tau_k - \tau_j)}}{c} \quad \text{for } k \geq 1.$$

The Jacobi determinant is given by

$$c(\tau_1, \dots, \tau_n) = \prod_{m=1}^n \left(a_{m0} + \sum_{j=1}^{m-1} a_{mj} e^{-c(\tau_m - \tau_j)} \right).$$

The joint density function in the region of $t_1 < \dots < t_n$ is given by

$$f(t_1, \dots, t_n) = c(t_1, \dots, t_n) e^{-\left(\sum_{l=1}^n a_{l0} t_l + \sum_{l=2}^n \sum_{j=1}^{l-1} a_{lj} \frac{1 - e^{-c(t_l - t_j)}}{c} \right)}. \quad (6)$$

The joint density function in other regions can be derived with permutation. We can then use it to find the distribution functions of ordered default times and other information. Note that the Markov chain approach (see Frey and Backhaus (2004), Herbertsson and Rootzen (2006)) can not deal with self-exciting point processes.

3 Factor Contagion Intensity Models

Suppose a portfolio of n obligors is composed of K subportfolios. There is a common factor Z which affects default probabilities of all obligors and given Z all subportfolios are independent of each other. The subportfolio i has n_i obligors. The default intensity of obligor j in subportfolio i is modelled by

$$\lambda_{ij}(t; Z) = a_i(Z) \left(1 + \sum_{l=1}^{n_i-1} c_i(Z) 1_{\{\tau_i^l \leq t\}} \right) \quad (7)$$

where τ_i^l is the l th default time in subportfolio i , and a_i and c_i are given functions of Z . The homogeneous contagion portfolio and the heterogeneous conditionally independent portfolio are two special cases with $K = 1$ and $K = n$, respectively.

This setup covers a wide range of applications. A portfolio (e.g., iTraxx Europe) may contain obligors from different sectors (e.g., autos, energy). All obligors are subject to the macro economic risk (e.g., interest rate, inflation) which is modelled with the common factor Z . Given Z , obligors in different

sectors are independent of each other but are correlated within a sector due to business link or competition. The standard conditional independence model assumes all obligors within a sector are independent of each other and ignores their interaction. This may underestimate the dependence risk and overestimate the portfolio value. The recent sub-prime mortgage debt crisis may be explained along this line of argument. Although the general relation of obligors within a sector can be heterogeneous, their true relation is not observable. The homogeneous contagion subportfolio assumption strikes a reasonable balance between the full generality of dependence structure and the easy tractability of model calibration.

Denote τ^k the k th default time and $N(t)$ the number of defaults by time t . Then $P(N(t) = k) = P(\tau^k \leq t < \tau^{k+1})$ and $P(\tau^k > t) = \sum_{i=0}^{k-1} P(N(t) = i)$. Denote $N_i(t)$ and $n_i(t)$ the number of defaults in subportfolios 1 to i and in subportfolio i by time t , respectively. Clearly, $N_1(t) = n_1(t)$, $N_{i+1}(t) = N_i(t) + n_{i+1}(t)$, and $N(t) = N_K(t)$. Suppose we know how to compute the conditional probabilities $P(n_i(t) = m|Z)$ for all i and m , given Z , then we can apply the recursive algorithm of Andersen et al. (2003) to find the conditional probabilities $P(N(t) = k|Z)$ as follows: We compute first $P(N_1(t) = k|Z) = P(n_1(t) = k|Z)$. Assume conditional probabilities $P(N_i(t) = m|Z)$ have been computed for $i \geq 1$ and $m \leq k$. The conditional independence of subportfolios 1 to i and subportfolio $i + 1$ implies that

$$P(N_{i+1}(t) = k|Z) = \sum_{m=0}^k P(n_{i+1}(t) = m|Z)P(N_i(t) = k - m|Z).$$

Repeat this process until we have computed $P(N_K(t) = k|Z)$ which equals $P(N(t) = k|Z)$ by construction. Finally, we can find the probability distribution of τ^k from the relation

$$P(\tau^k > t) = \sum_{i=0}^{k-1} E[P(N(t) = i|Z)].$$

From the discussion above it is clear that we only need to find conditional probabilities $P(n_i(t) = m|Z)$ in order to find probabilities $P(\tau^k \leq t)$ and $P(N(t) = k)$ for general factor contagion portfolios. Since Z is given, we may simply omit it and focus on instead the following homogeneous intensity processes:

$$\lambda_i(t) = a \left(1 + \sum_{j=1, j \neq i}^n c 1_{\{\tau_j \leq t\}} \right), \quad i = 1, \dots, n, \quad (8)$$

where a and c are constant. If $c = 0$ then defaults of names are independent. If $c > 0$ then default of one name increases default intensity of other names by a factor of c . The next result characterizes the probability distributions of τ^k and $N(t)$.

Lemma 1 *Assume that $c \neq 1/i$ for $i = 1, \dots, n-1$. Then the density function of τ^k is given by*

$$f_{\tau^k}(t) = \sum_{j=0}^{k-1} \alpha_{k,j} a e^{-\beta_j a t} \quad (9)$$

where

$$\begin{aligned} \alpha_{k,j} &= \frac{(-1)^{k-1-j} n! \left(\prod_{m=1}^{k-1} (1 + mc) \right)}{(n-k)! j! (k-1-j)! \left(\prod_{m=0, m \neq j}^{k-1} (1 + (m+j-n)c) \right)} \quad (10) \\ \beta_j &= (n-j)(1+jc). \end{aligned}$$

The probability distribution of $N(t)$ is given by

$$P(N(t) = k) = \sum_{j=0}^k \frac{(-1) \alpha_{k,j}}{(k-j)(1+(k+j-n)c)} e^{-\beta_j a t}. \quad (11)$$

For $k = 1$ we have $f_{\tau^1}(t) = nae^{-nat}$ which shows contagion has no effect on the first default time as expected. For $k = 2$ we have $f_{\tau^2}(t) = \frac{n(n-1)(1+c)a}{(1+(1-n)c)} (-e^{-nat} + e^{-(n-1)(1+c)at})$, provided $c \neq 1/(n-1)$, which implies contagion affects the second and all subsequent default times.

Lemma 1 characterizes the density function of τ^k when $c \neq 1/i$ for $i = 1, \dots, n-1$. If $c = 1/i$ then some factors in the denominators of $\alpha_{k,j}$ may become zero and $f_{\tau^k}(t)$ is not well defined. The following result modifies Lemma 1 to accommodate that case.

Lemma 2 *Assume that $c = 1/i_0$ for some $i_0 \in \{1, \dots, n-1\}$. Let $m_0 = n - i_0$. If $m_0 > 2k - 2$ then $\alpha_{k,j}$ are given by (10). If $m_0 \leq 2k - 2$ then*

$$\alpha_{k,j} = \frac{(-1)^{k-1-j} n! \left(\prod_{m=1}^{k-1} (1 + mc) \right) \left(\sum_{m=0, m \neq j, m_0-j}^{k-1} \frac{1}{(m-j)c} - (n-j)at \right)}{(n-k)! j! (k-1-j)! \left(\prod_{m=0, m \neq j, m_0-j}^{k-1} (1 + (m+j-n)c) \right)} \quad (12)$$

for $\max(m_0 - k + 1, 0) \leq j \leq \min(m_0, k-1)$ except $j = m_0/2$ (when m_0 is an even number) and the remaining $\alpha_{k,j}$ are given by (10).

Note that coefficients $\alpha_{k,j}$ depend on time t in Lemma 2. For $k = 2$ and $c = 1/(n - 1)$, we have $i_0 = n - 1$, $m_0 = 1$, and $m_0 \in [0, 2k - 2]$, $\alpha_{2,j}$ for $j = 0, 1$ are therefore computed from (12). We have in this case $f_{\tau^2}(t) = (na)^2 te^{-nat}$.

We can now price the k th-to-default basket CDS. Assume S_k is the annualized swap rate, paid at time t_i , $i = 1, \dots, N$, where $0 = t_0 < t_1 < \dots < t_N = T$ and T is the maturity of the contract, $\Delta_i = t_i - t_{i-1}$, R is recovery rate, and r is the riskless interest rate. The value of the contingent leg at time 0 is equal to

$$C_k = (1 - R)e^{-r\tau^k} 1_{\{\tau^k \leq T\}}$$

and the value of the fee leg at time 0 is equal to

$$F_k = \sum_{i=1}^N \left[S_k \Delta_i e^{-rt_i} 1_{\{\tau^k > t_i\}} + S_k (\tau^k - t_{i-1}) e^{-r\tau^k} 1_{\{t_{i-1} < \tau^k \leq t_i\}} \right].$$

If the underlying portfolio is a homogeneous contagion one, we can easily compute the expectations of C_k and F_k with the density function $f_{\tau^k}(t)$ and find the k th default swap rate S_k . Here is the result:

Theorem 1 *Let the intensity processes $\lambda_i(t)$ be given by (8) and $c \neq 1/i$ for $i = 1, \dots, n - 1$. Then*

$$\begin{aligned} E(C_k) &= (1 - R) \sum_{j=0}^{k-1} \frac{\alpha_{k,j} a}{r + \beta_j a} (1 - e^{-(r + \beta_j a)T}) \\ E(F_k) &= S_k \sum_{i=1}^N \sum_{j=0}^{k-1} \frac{\alpha_{k,j} a}{(r + \beta_j a)^2} \left[e^{-(r + \beta_j a)t_{i-1}} + \left(\frac{\Delta_i r (r + \beta_j a)}{\beta_j a} - 1 \right) e^{-(r + \beta_j a)t_i} \right] \end{aligned}$$

where $\alpha_{k,j}$ and β_j are given by (10). The k th default swap rate S_k is obtained by equating $E(C_k)$ and $E(F_k)$.

Theorem 1 gives the analytic pricing formula for basket CDS rates at time 0. It can also be applied to give the k th default swap rate $S_k(t)$ at any time t by replacing t_i by $t_i - t$, where t_i are all remaining swap payment dates after t . If there are already m defaults at time t ($0 \leq m < k$) then k is replaced by $k - m$.

For general factor contagion portfolios modelled by intensity processes (7) there is no simple analytic formula as in Theorem 1 for basket swap rates, but we can find them easily with the recursive algorithm.

4 Counterparty Risk and Stochastic Intensity

In this section we show that the results of the last section can be extended to other contagion intensity processes to accommodate a wide range of models in portfolio credit risk, including the counterparty risk and the stochastic intensity process.

Assume the intensity processes λ_i are given by (8). Since $\tau^1 < \tau^2 < \dots < \tau^n$ the joint density function of $\tau = (\tau^1, \dots, \tau^n)$ is given by

$$f_\tau(t_1, \dots, t_n) = n!Ce^{-(w_1t_1 + \dots + w_nt_n)}$$

for $0 < t_1 < \dots < t_n$ and 0 for all other $t_1, \dots, t_n$, where $C = a^n \prod_{m=1}^{n-1} (1 + mc)$. Suppose the default time τ^B of the counterparty is modelled by the intensity process

$$\lambda_B(t) = b_0 + \sum_{i=1}^n b_i 1_{\{\tau^i \leq t\}}. \quad (13)$$

Note that the hazard rate process $\lambda_B(t)$ of the counterparty is influenced by defaults of underlying names i , but not vice versa. This follows the observations in Leung and Kwok (2005) and Yu (2007) that the contagion of seller B on underlying names i does not have impact on CDS pricing.

We now describe the joint distribution of (τ^k, τ^B) for $k = 1, \dots, n$.

Lemma 3 *Let the intensity processes $\lambda_i(t)$ and $\lambda_B(t)$ be given by (8) and (13) respectively, and $c \neq 1/i$ for $i = 1, \dots, n-1$. Then the joint density functions of (τ^k, τ^B) , $k = 1, \dots, n$, are given by*

$$f_{\tau^k, \tau^B}(t_k, t_B) = \begin{cases} \sum_{j=k}^n \sum_{j_1=0}^{k-1} \alpha_{k,j,j_1} e^{-\beta_{j,j_1} t_B - \gamma_{j,j_1} t_k} & \text{if } t_k < t_B \\ \sum_{j=0}^{k-1} \sum_{j_1=0}^j \tilde{\alpha}_{k,j,j_1} e^{-\tilde{\beta}_{j,j_1} t_B - \tilde{\gamma}_{j,j_1} t_k} & \text{if } t_k > t_B \end{cases} \quad (14)$$

where $\beta_{j,j_1} = \sum_{l=0}^j b_l + \beta_j a$, $\gamma_{j,j_1} = \beta_{j_1} a - \beta_j a$, $\tilde{\beta}_{j,j_1} = \sum_{l=0}^{j_1} b_l + \beta_{j_1} a - \beta_j a$, $\tilde{\gamma}_{j,j_1} = \beta_j a$, $\beta_j = (n-j)(1+jc)$, and α_{k,j,j_1} , $\tilde{\alpha}_{k,j,j_1}$ are given by (25), (26).

The slightly involved expression for f_{τ^k, τ^B} in (14) is due to the assumption that the hazard rate process $\lambda_B(t)$ depends on default times of names i , $i = 1, \dots, n$. For example, to compute the joint probability $P(\tau^k > t_k, \tau^B > t_B)$ for $t_k > t_B$ we need to know how many defaults have occurred before time t_B to determine uniquely the hazard rate process $\lambda_B(t)$ and we have to compute probabilities of k mutually independent events $\{\tau^k > t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B\}$, $i = 0, \dots, k-1$, with $\tau^0 = 0$.

A straightforward integration, using the joint density function f_{τ^k, τ^B} in (14), shows that

$$\begin{aligned} P(\tau^k > t_i, \tau^B > t_i) &= \sum_{j=k}^n \sum_{j_1=0}^{k-1} \frac{\alpha_{k,j,j_1}}{\gamma_{j,j_1}} \cdot \frac{e^{-(\beta_{j,j_1} + \gamma_{j,j_1})t_i}}{\beta_{j,j_1} + \gamma_{j,j_1}} \\ &\quad + \sum_{j=0}^{k-1} \sum_{j_1=0}^j \frac{\tilde{\alpha}_{k,j,j_1}}{\tilde{\gamma}_{j,j_1}} \cdot \frac{e^{-(\tilde{\beta}_{j,j_1} + \tilde{\gamma}_{j,j_1})t_i}}{\tilde{\beta}_{j,j_1} + \tilde{\gamma}_{j,j_1}}. \end{aligned} \quad (15)$$

We can now price the k th default swap rate. The value of the contingent leg at time 0 is equal to

$$C_k = (1 - R)e^{-r\tau^k} 1_{\{\tau^k \leq T, \tau^B \geq \tau^k\}}$$

and the value of the fee leg at time 0 (assuming there is no accrued interest) is equal to

$$F_k = \sum_{i=1}^N S_k \Delta_i e^{-rt_i} 1_{\{\tau^k > t_i, \tau^B > t_i\}}.$$

Computing the expectations of C_k and F_k with the density function (14) we derive the following result for the k th default swap rate S_k :

Theorem 2 *Let the intensity processes $\lambda_i(t)$ and $\lambda_B(t)$ be given by (8) and (13) respectively, and $c \neq 1/i$ for $i = 1, \dots, n-1$. Then*

$$\begin{aligned} E(C_k) &= (1 - R) \sum_{j=k}^n \sum_{j_1=0}^{k-1} \frac{\alpha_{k,j,j_1}}{\beta_{j,j_1}} \cdot \frac{1 - e^{-(r + \beta_{j,j_1} + \gamma_{j,j_1})T}}{r + \beta_{j,j_1} + \gamma_{j,j_1}} \\ E(F_k) &= S_k \sum_{i=1}^N \Delta e^{-rt_i} P(\tau^k > t_i, \tau^B > t_i) \end{aligned}$$

where $P(\tau^k > t_i, \tau^B > t_i)$ is given by (15). The k th default swap rate S_k is obtained by equating $E(C_k)$ and $E(F_k)$.

The homogeneous contagion intensity processes can be generalized to the case in which the constant intensity rate a is replaced by some stochastic process X , i.e.,

$$\lambda_i(t) = X(t) \left(1 + \sum_{j=1, j \neq i}^n c 1_{\{\tau_j \leq t\}} \right). \quad (16)$$

The affine jump diffusion process is one of the most useful and tractable stochastic processes. It has been applied in modeling and pricing bonds, options, credit derivatives, etc., see the survey article (Duffie (2005)) and the technical papers (Duffie et al. (2000), Duffie et al. (2003)) and references within. Let X be a Markov state process in R^1 satisfying the stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) + dZ(t) \quad (17)$$

where W is a standard Brownian motion, $\mu, \sigma : R^1 \rightarrow R^1$, and Z is a pure jump process whose jumps have a fixed probability distribution ν and arrival intensity $\{\lambda(X(t)) : t \geq 0\}$ for some $\lambda : R^1 \rightarrow R^1$. X has an infinitesimal generator $\mathcal{D}$ of the Levy type, defined by

$$\mathcal{D}f(x) = f'(x)\mu(x) + \frac{1}{2}f''(x)\sigma^2(x) + \lambda(x) \int_{R^1} (f(x+z) - f(x))d\nu(z)$$

where f is a bounded C^2 function with bounded first and second derivatives. Assume that μ, σ^2 , and λ are affine functions, i.e., $\mu(x) = \mu_0 + \mu_1x$, $\sigma^2(x) = \sigma_0 + \sigma_1x$, and $\lambda(x) = \lambda_0 + \lambda_1x$. Then

$$E \left(e^{-\int_t^T RX(s)ds} | \mathcal{F}_t \right) = e^{A(T-t) + B(T-t)X(t)} \quad (18)$$

where R is a positive constant and A, B satisfy the ODEs

$$\begin{aligned} \dot{A}(t) &= \mu_0 B(t) + \frac{1}{2} \sigma_0 B^2(t) + \lambda_0 (\theta(B(t)) - 1) \\ \dot{B}(t) &= -R + \mu_1 B(t) + \frac{1}{2} \sigma_1 B^2(t) + \lambda_1 (\theta(B(t)) - 1) \end{aligned} \quad (19)$$

with initial conditions $A(0) = 0$ and $B(0) = 0$, where $\theta(c) = \int_{R^1} e^{cz} d\nu(z)$. Some “well behaved” conditions on μ, σ^2, λ are needed to ensure all integrals involved are well defined and the ODEs (19) admit a unique solution, see Duffie et al. (2000), page 1351, for details. If X follows the basic affine process

$$dX(t) = k(\bar{x} - X(t))dt + \sigma\sqrt{X(t)}dW(t) + dZ(t),$$

where $Z(t)$ is a compound Poisson process with Poisson arrival intensity λ and exponential jump size with mean γ and all parameters are positive constant. Then equations (19) are simplified to

$$\begin{aligned} \dot{A}(t) &= k\bar{x}B(t) + \frac{\lambda\gamma B(t)}{1 - \gamma B(t)} \\ \dot{B}(t) &= -R - kB(t) + \frac{1}{2}\sigma^2 B^2(t). \end{aligned} \quad (20)$$

Duffie and Garleanu (2001) show that there is a closed-form solution to equation (20). The solution $B(t)$ to the Riccati equation is given by

$$B(t) = \frac{1 - e^{b_1 t}}{c_1 + d_1 e^{b_1 t}}$$

where

$$\begin{aligned} c_1 &= \frac{k + \sqrt{k^2 + 2\sigma^2 R}}{-2R} \\ d_1 &= \frac{k}{R} + c_1 \\ b_1 &= \frac{k d_1 + 2R c_1 d_1 - k c_1 + \sigma^2}{c_1 + d_1}. \end{aligned}$$

In particular, if $\sigma = 0$ then $B(t) = -\frac{R}{k}(1 - e^{-kt})$. Once $B(t)$ is determined $A(t)$ can be found in a closed-form, see Duffie and Garleanu (2001) for its expression and other details. By differentiating with respect to T to equation (18) we have

$$E \left(X(T) e^{-\int_t^T R X(s) ds} | \mathcal{F}_t \right) = -\frac{1}{R} e^{A(T-t) + B(T-t)X(t)} (\dot{A}(T-t) + \dot{B}(T-t)x(t)).$$

The next result characterizes the conditional density function of τ^k given the complete sample path information $\mathcal{F}_\infty^X$.

Lemma 4 *Assume that $c \neq 1/i$ for $i = 1, \dots, n-1$. The density function of τ^k , given $\mathcal{F}_\infty^X$, is given by*

$$f_{\tau^k}(t) = \sum_{j=0}^{k-1} \alpha_{k,j} X(t) e^{-\beta_j \int_0^t X(u) du}$$

where $\alpha_{k,j}$ and β_j are given by (10). The expectation of $f_{\tau^k}(t)$ is given by

$$E[f_{\tau^k}(t)] = \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{-\beta_j} H_j(t, X(0))$$

where

$$H_j(t, x) = e^{A_j(t) + B_j(t)x} (\dot{A}_j(t) + \dot{B}_j(t)x)$$

and A_j and B_j are the solutions to equation (19) with $R = \beta_j$.

	c	0		0.3		3	
a	k	MC	AP	MC	AP	MC	AP
0.01	1	0.5067	0.5058	0.5058	0.5058	0.5067	0.5058
	2	0.2266	0.2263	0.2631	0.2630	0.3953	0.3948
	5	0.0296	0.0295	0.0848	0.0847	0.3028	0.3024
	10	0.0001	0.0001	0.0136	0.0135	0.2531	0.2527
	20	0.0000	0.0000	0.0001	0.0001	0.2115	0.2112
	50	0.0000	0.0000	0.0000	0.0000	0.1562	0.1561
	100	0.0000	0.0000	0.0000	0.0000	0.0121	0.0121
0.1	1	5.0364	5.0242	5.0364	5.0242	5.0364	5.0242
	2	2.5012	2.4984	2.8290	2.8254	4.0158	4.0082
	5	0.9801	0.9800	1.4614	1.4607	3.1764	3.1719
	10	0.4714	0.4713	0.9621	0.9616	2.7362	2.7326
	20	0.2036	0.2039	0.6687	0.6686	2.3816	2.3790
	50	0.0000	0.0000	0.4225	0.4224	1.9528	1.9509
	100	0.0000	0.0000	0.1203	0.1206	1.1451	1.1649

Table 1: Basket CDS rates with data $n = 100$, $T = 3$, $\Delta = 0.5$, $r = 0.05$, and $R = 0.5$.

A straightforward integration, using the law of iterated expectation (on $\mathcal{F}_\infty^X$) and the density function $f_{\tau^k}(t)$, shows that the k th default swap rate S_k is determined by the relation $E(C_k) = E(F_k)$, where

$$\begin{aligned}
E(C_k) &= (1 - R) \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{-\beta_j} \int_0^T e^{-rt} H_j(t, X(0)) dt \\
E(F_k) &= S_k \sum_{i=1}^N \sum_{j=0}^{k-1} \Delta_i e^{-rt_i} \frac{\alpha_{k,j}}{-\beta_j} \int_{t_i}^\infty H_j(t, X(0)) dt.
\end{aligned}$$

It is easy to find basket CDS rates at time t , which depend on $X(t)$ and the number of defaults occurred by time t , see the discussion after Theorem 1. We can also introduce common factors Z into the dynamics of $X(t)$.

5 Numerical Issues

We have done some numerical tests with both the analytic pricing method (AP) and the Monte Carlo method (MC). Table 1 lists the swap rates of a homogeneous portfolio with the following data: number of obligors $n = 100$,

k	S_k	c	a	c	a	c	a
2	2.4984	0	0.1	0.3	0.0884	3	0.0622
10	0.4713	0	0.1	0.3	0.0496	3	0.0175
20	0.2039	0	0.1	0.3	0.0338	3	0.0097
30	0.0332	0	0.1	0.3	0.0246	3	0.0048
40	0.0002	0	0.1	0.3	0.0189	3	0.0029

Table 2: Basket CDS rates with data $n = 100$, $T = 3$, $\Delta = 0.5$, $r = 0.05$, $R = 0.5$, and various parameters a and c .

time to maturity $T = 3$, risk free interest rate $r = 0.05$, recovery rate $R = 0.5$, semiannual payment $\Delta t = 0.5$, and varying default intensity rates a and contagion rates c . It is clear that both methods produce the same results. Contagion has great impact on swap rates, which becomes increasingly significant as k increases. If there is no contagion ($c = 0$) then defaults of names are independent and swap rates decrease fast as k increases. If there is contagion (e.g., $c = 3$) then swap rates decrease slowly as k increases, which indicates that one default is likely to trigger “infectious defaults” of others. The analytic method takes less than 8 seconds to compute all swap rates with MAPLE on an Intel 1.83GHz laptop, whereas the Monte Carlo method takes more than 12 minutes to complete for 100,000 simulation runs.

Table 2 illustrates the effect of contagion on CDS rates. We adopt the approach of Davis and Lo (2001) in which we fix the swap rates S_k and increase the contagion rate c and then find the corresponding default intensity rate a . It is clear that as c increases a decreases for all swap rates, but the magnitude is different. When k is small, the decrease of a is moderate and the contagion effect of c is relatively small. However, when k is large, the decrease of a is significant and the contagion effect of c is very pronounced. Table 2 implies that the k th default swap rates are sensitive to the contagion effect when k is large, in other words, even underlying names have low default probabilities, investors of “super senior tranches” may suffer heavy losses due to the contagion effect.

Table 3 displays the swap rates of a portfolio of 150 obligors, which consists of 3 independent homogeneous subportfolios of 50 obligors each with the same default intensity rate 0.01. The other data are the same as those in Table 1. Columns 2-3 correspond to contagion rates $c_1 = c_2 = c_3 = 0.3$, columns 4-5 to $c_1 = c_2 = c_3 = 3$, and columns 6-7 to $c_1 = 30$, $c_2 = 3$, $c_3 = 0.3$. The Monte Carlo method and the analytic method produce the same results. The swap rates are much higher in columns 4-5 than those in

k	MC	AP	MC	AP	MC	AP
1	0.7585	0.7583	0.7585	0.7582	0.7585	0.7582
2	0.3818	0.3821	0.4945	0.4945	0.4960	0.4960
10	0.0148	0.0149	0.2220	0.2219	0.2855	0.2847
50	0.0000	0.0000	0.0441	0.0439	0.1891	0.1888
51	0.0000	0.0000	0.0418	0.0417	0.1644	0.1643
52	0.0000	0.0000	0.0397	0.0396	0.1389	0.1389
60	0.0000	0.0000	0.0256	0.0255	0.0639	0.0642
100	0.0000	0.0000	0.0009	0.0009	0.0002	0.0002
101	0.0000	0.0000	0.0008	0.0008	0.0001	0.0001
110	0.0000	0.0000	0.0002	0.0003	0.0000	0.0000

Table 3: Basket CDS rates with data $n_1 = n_2 = n_3 = 50$, $a_1 = a_2 = a_3 = 0.01$, $T = 3$, $\Delta = 0.5$, $r = 0.05$, and $R = 0.5$. Columns 2-3 correspond to $c_1 = c_2 = c_3 = 0.3$, columns 4-5 to $c_1 = c_2 = c_3 = 3$ and columns 6-7 to $c_1 = 30$, $c_2 = 3$, $c_3 = 0.3$.

columns 2-3 to reflect the increased contagion risk. It is interesting to note that the swap rates in columns 6-7 decrease slowly for the first 50 names and then decrease quickly afterwards, which implies that the high contagion rate c_1 is likely to trigger “infectious defaults” of other names in subportfolio 1.

The analytic pricing formula involves the computation of combinatorial numbers. When n and k are large it may exceed the limit of numerical accuracy of a computer and overflow/underflow errors may develop. We may alleviate the problem by computing $\alpha_{k,j}$ recursively with the relation

$$\alpha_{k,j} = c_{k,j} \alpha_{k,j-1}$$

for $j = 1, \dots, k-1$, where

$$c_{k,j} = \frac{(-1)(k-j)(1+(2j-n)c)(1+(j-n-1)c)}{j(1+(2j-n-2)c)(1+(k+j-n-1)c)}.$$

and

$$\alpha_{k,0} = \frac{(-1)^{k-1} n!}{(n-k)!(k-1)!} \prod_{m=1}^{k-1} \frac{(1+mc)}{(1+(m-n)c)}.$$

In fact, there is no need to compute $\alpha_{k,0}$ as it is the common factor of both $E(C_k)$ and $E(F_k)$, and is cancelled out in computation.

We have used C++, Fortran, and MAPLE to test our numerical example. We find that MAPLE has the best performance (always accurate even for

large n , e.g., $n = 250$), Fortran is the second best (accurate for moderate n with real*16 data type), and C++ is the worst (accurate only for small n with double data type). For example, if $n = 100$ with $a = 0.1$ and $c = 0.3$, c++ produces correct values for k up to 25 and then becomes unstable, Fortran gives correct values for k up to 49 and then becomes unstable, however, MAPLE computes correct values for all n , see Table 1. This is due to MAPLE uses the exact algorithm and one can specify the number of decimal points to be used in computation. For example, in Table 1, we use 100 decimal point accuracy in MAPLE computation. In comparison, Fortran and C++ use the floating-point algorithm and one can only choose the data type which has only limited number of decimal points of accuracy. When n and k are large, default probabilities are very small and numerical errors propagate, which eventually go beyond the scope of numerical accuracy of the compilers.

One way to deal with large n and k is that we work on probabilities instead of direct application of the analytic formula. For example, to find $E(C_k)$, we can use the formula

$$E(C_k) = (1 - R) \left(e^{-rT} P(\tau^k \leq T) + r \int_0^T P(\tau^k \leq t) e^{-rt} dt \right),$$

which is the sum of positive terms, and then find the numerical integration value. The reason for this approach is that $P(\tau^k \leq t)$ is a probability and is a decreasing sequence of k for fixed t . If it becomes negative or greater than the previous value, then something is wrong numerically and an adjustment can be made at spot. This flexibility is lost with the analytic formula which has no clear probability interpretation. With this amendment we can produce the same results in Table 1 with C++. The same technique is used to get the analytic results in Table 3 with C++.

6 Conclusions

In this paper we propose a factor contagion model for correlated defaults, which unifies the factor model and the contagion model approaches. The portfolio is made up of subportfolios which are conditionally independent of each other and defaults of obligors in each subportfolio are driven by infectious hazard rate processes. The model covers the heterogeneous conditionally independent portfolio and the factor infectious default portfolio as special cases.

We use the total hazard construction method to derive the joint distribution of default times and the distribution of ordered default times for

homogeneous contagion portfolios, which makes easy in pricing portfolio credit derivatives and in measuring portfolio credit risk such as VaR and conditional VaR. We give the analytic basket CDS pricing formula for homogeneous contagion portfolios and suggest a recursive algorithm for general factor contagion portfolios. We extend the results to include the counterparty risk and the stochastic intensity process.

The key advantage of the analytic pricing formula is that it is hugely fast in pricing and provides an amenable way for greeks and calibration. We compare numerically the analytic results with the Monte Carlo results and show that the analytic pricing formula gives correct values for large portfolios with MAPLE, but may become numerically unstable with C++ or Fortran, which is purely due to the floating-point algorithm and the limited numerical accuracy of the compilers. We suggest an alternative way of implementation which removes the numerical instability problem and gives the correct values for large portfolios with C++.

7 Proofs of Lemmas

In the proofs of lemmas we need to evaluate the following integral

$$A_k(t, w_1, \dots, w_k) := \int_0^t \int_{t_1}^t \dots \int_{t_{k-1}}^t e^{-(w_1 t_1 + \dots + w_k t_k)} dt_1 \dots dt_k$$

where $w_1, \dots, w_k$ are constants. Integration with respect to dt_k gives the recursive formula

$$\begin{aligned} A_k(t, w_1, \dots, w_k) &= -\frac{1}{w_k} e^{-w_k t} A_{k-1}(t, w_1, \dots, w_{k-1}) \\ &\quad + \frac{1}{w_k} A_{k-1}(t, w_1, \dots, w_{k-2}, w_{k-1} + w_k) \end{aligned} \quad (21)$$

The induction method shows

$$A_k(t, w_1, \dots, w_k) = \sum_{j=0}^k \gamma_{k,j}(w_1, \dots, w_k) e^{-(\sum_{l=j+1}^k w_l)t} \quad (22)$$

where

$$\gamma_{k,j}(w_1, \dots, w_k) = \frac{(-1)^{k-j}}{\left\{ \prod_{m=j+1}^k (\sum_{l=j+1}^m w_l) \right\} \cdot \left\{ \prod_{m=1}^j (\sum_{l=m}^j w_l) \right\}} \quad (23)$$

with the convention that $\prod_{i=j}^k a_i = 1$ and $\sum_{i=j}^k a_i = 0$ if $k < j$. In fact, (22) trivially holds for $k = 1$. Assume (22) holds for $k - 1$ where $k \geq 2$. Substituting (22) into (21) and combining the first $k - 1$ (i.e., $j = 0, \dots, k - 2$) terms together, we can show that (22) holds for k too.

Here we have implicitly assumed that all factors in the denominator of $\gamma_{k,j}$ are nonzero. If for some $j_0 + 1 \leq m_0 \leq k$ the factor $\sum_{l=j_0+1}^{m_0} w_l = 0$, then γ_{k,j_0} and γ_{k,m_0} are the only two terms containing $\sum_{l=j_0+1}^{m_0} w_l$ in their denominators and are not defined. However, we can show with the L'Hôpital's rule that there exists a limit for $\gamma_{k,j_0} e^{-(\sum_{l=j_0+1}^k w_l)t} + \gamma_{k,m_0} e^{-(\sum_{l=m_0+1}^k w_l)t}$ as $\sum_{l=j_0+1}^{m_0} w_l \rightarrow 0$. We can replace the sum of the undefined terms with the corresponding limit, see the details in the proof of Lemma 2. Note also that by changes of variables we have

$$\int_{\alpha}^t \int_{t_1}^t \dots \int_{t_{k-1}}^t e^{-(w_1 t_1 + \dots + w_k t_k)} dt_1 \dots dt_k = e^{-(w_1 + \dots + w_k)\alpha} A_k(t - \alpha, w_1, \dots, w_k)$$

and by differentiation and simplification we have

$$\frac{d}{dt} A_k(t, w_1, \dots, w_k) = e^{-w_k t} A_{k-1}(t, w_1, \dots, w_{k-1}).$$

Proof of Lemma 1:

$$\begin{aligned} & P(\tau_1 < \dots < \tau_k < t < \tau_{k+1} < \dots < \tau_n) \\ &= \int_0^t \int_{t_1}^t \dots \int_{t_{k-1}}^t \int_t^\infty \int_{t_{k+1}}^\infty \dots \int_{t_{n-1}}^\infty f(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= C \left(\int_0^t \int_{t_1}^t \dots \int_{t_{k-1}}^t e^{-w_1 t_1 - \dots - w_k t_k} dt_1 \dots dt_k \right) \\ & \quad \cdot \left(\int_t^\infty \int_{t_{k+1}}^\infty \dots \int_{t_{n-1}}^\infty e^{-w_{k+1} t_{k+1} - \dots - w_n t_n} dt_{k+1} \dots dt_n \right) \\ &= C \left(\sum_{j=0}^k \gamma_{k,j}(w_1, \dots, w_k) e^{-(\sum_{l=j+1}^k w_l)t} \right) \left(\frac{1}{\prod_{m=k+1}^n (\sum_{l=m}^n w_l)} e^{-(\sum_{l=k+1}^n w_l)t} \right) \\ &= \sum_{j=1}^k \frac{(-1)^{k-j} \prod_{m=1}^{k-1} (1 + mc)}{(k-j)! j! (n-k)! \prod_{m=0, m \neq j}^k (1 - (n-m-j)c)} e^{-(n-j)(1+jc)at}. \end{aligned}$$

Here we have used (22) and the relations

$$C = a^n \prod_{m=1}^{n-1} (1 + mc)$$

$$\begin{aligned}
w_l &= a(1 - (n - 2l + 1)c) \\
\sum_{l=n_1}^{n_2} w_l &= (n_2 - n_1 + 1)a(1 - (n + 1 - n_1 - n_2)c) \\
\prod_{m=j+1}^k \left(\sum_{l=j+1}^m w_l \right) &= (k - j)!a^{k-j} \prod_{m=j+1}^k (1 - (n - m - j)c) \\
\prod_{m=1}^j \left(\sum_{l=m}^j w_l \right) &= j!a^j \prod_{m=0}^{j-1} (1 - (n - m - j)c) \\
\prod_{m=k+1}^n \left(\sum_{l=m}^n w_l \right) &= (n - k)!a^{n-k} \prod_{m=k}^{n-1} (1 + mc)
\end{aligned}$$

The homogeneous and symmetric property implies that

$$P(\tau^k \leq t < \tau^{k+1}) = n!P(\tau_1 < \dots < \tau_k < t < \tau_{k+1} < \dots < \tau_n),$$

which gives (11). We now compute the k th default time distribution

$$\begin{aligned}
&P(\tau^k > t) \\
&= \sum_{i=0}^{k-1} P(\tau^i < t, \tau^{i+1} > t) \\
&= \sum_{i=0}^{k-1} n! \sum_{j=0}^i \frac{(-1)^{i-j} \prod_{m=1}^{i-1} (1 + mc)}{(i - j)!j!(n - i)! \prod_{m=0, m \neq j}^i (1 - (n - m - j)c)} e^{-\beta_j at} \\
&= \sum_{j=0}^{k-1} \left\{ \sum_{i=j}^{k-1} \frac{(-1)^{i-j} \prod_{m=1}^{i-1} (1 + mc)}{(i - j)!(n - i)! \prod_{m=0, m \neq j}^i (1 - (n - m - j)c)} \right\} \frac{n!}{j!} e^{-\beta_j at} \\
&= \sum_{j=0}^{k-1} \left\{ \sum_{i=0}^{k-1-j} \frac{(-1)^i \prod_{m=1}^{i+j-1} (1 + mc)}{i!(n - i - j)! \prod_{m=0, m \neq j}^{i+j} (1 - (n - m - j)c)} \right\} \frac{n!}{j!} e^{-\beta_j at} \\
&= \sum_{j=0}^{k-1} \left\{ \frac{(-1)^{k-1-j} \prod_{m=1}^{k-1} (1 + mc)}{(k - 1 - j)!(n - j)(n - k)!(1 + jc) \prod_{m=0, m \neq j}^{k-1} (1 - (n - m - j)c)} \right\} \frac{n!}{j!} e^{-\beta_j at}.
\end{aligned}$$

Here we have used the relation

$$\begin{aligned}
&\sum_{i=0}^l \frac{(-1)^i \prod_{m=1}^{i+j-1} (1 + mc)}{i!(n - i - j)! \prod_{m=0, m \neq j}^{i+j} (1 - (n - m - j)c)} \\
&= \frac{(-1)^l \prod_{m=1}^{l+j} (1 + mc)}{l!(n - j)(n - j - l - 1)!(1 + jc) \prod_{m=0, m \neq j}^{l+j} (1 - (n - m - j)c)}
\end{aligned}$$

which can be proved with the induction method. Differentiating $P(\tau^k > t)$ with respect to t leads to the density function of τ^k . $\square$

Proof of Lemma 2. Define

$$j_0 = \min\{j : 0 \leq j \leq k-1, j+m = m_0 \text{ for some } 0 \leq m \leq k-1\}.$$

Since $0 \leq j+m \leq 2k-2$ we know that if $m_0 > 2k-2$ then no such j_0 exists, which implies that no factors in the denominators of $\alpha_{k,j}$, $j = 0, \dots, k-1$, are zero and (9) is well defined. Now assume $0 \leq m_0 \leq 2k-2$. If $0 \leq m_0 \leq k-1$ then $j_0 = 0$ and $m = m_0$. If $k \leq m_0 \leq 2k-2$ then $j_0 = m_0 - k + 1$ and $m = k-1$. This is because if $m < k-1$ then we can increase m by 1 and decrease j_0 by 1 to get a feasible pair $(j_0 - 1, m+1)$ which contradicts j_0 being the smallest value.

Consider first the case when m_0 is an odd number. If $0 \leq m_0 \leq k-1$ then the feasible pairs (j, m) which makes $\alpha_{k,j}$ undefined are $(0, m_0), (1, m_0-1), \dots, (m_0, 0)$. We can write $f_{\tau^k}(t)$ of (9) as

$$f_{\tau^k}(t) = \sum_{j=0}^{(m_0-1)/2} \left(\alpha_{k,j} a e^{-\beta_j a t} + \alpha_{k, m_0-j} a e^{-\beta_{m_0-j} a t} \right) + \sum_{j=m_0+1}^{k-1} \alpha_{k,j} a e^{-\beta_j a t}.$$

The second term is well defined. However, both $\alpha_{k,j}$ and α_{k, m_0-j} have zero factors in their denominators and the first term is not defined. Perturb a to $(1+x)a$ for x near zero and define functions

$$\begin{aligned} \alpha_{k,j}(x) &= \frac{(-1)^{k-1-j} n! \prod_{m=1}^{k-1} (x+1+mc)}{(n-k)! j! (k-1-j)! x \prod_{m \neq j, m_0-j} (x+1+(m+j-n)c)} \\ \beta_j(x) &= (n-j)(x+1+jc) \\ \alpha_{k, m_0-j}(x) &= \frac{(-1)^{k-1-m_0+j} n! \prod_{m=1}^{k-1} (x+1+mc)}{(n-k)! (m_0-j)! (k-1-m_0+j)! x \prod_{m \neq j, m_0-j} (x+1+(m+m_0-j-n)c)} \\ \beta_{m_0-j}(x) &= (n-m_0+j)(x+1+(m_0-j)c). \end{aligned}$$

If we write $\alpha_{k,j}(x) e^{-\beta_j(x)at} + \alpha_{k, m_0-j}(x) e^{-\beta_{m_0-j}(x)at}$ as a rational function of x , also note that $1 = (n-m_0)c$ and $\beta_{m_0-j} = (n-m_0+j)(1+(m_0-j)c) = (n-m_0+j)(n-j)c = (1+jc)(n-j) = \beta_j$, then a lengthy but straightforward calculation shows that both numerator and denominator tend to zero as $x \rightarrow 0$. We can then apply the L'Hôpital's rule to show that

$$\lim_{x \rightarrow 0} \left(\alpha_{k,j}(x) e^{-\beta_j(x)at} + \alpha_{k, m_0-j}(x) e^{-\beta_{m_0-j}(x)at} \right) = \hat{\alpha}_{k,j} e^{-\beta_j at} + \hat{\alpha}_{k, m_0-j} e^{-\beta_{m_0-j} at}$$

where $\hat{\alpha}_{k,j}$ is defined by (12).

If $k \leq m_0 \leq 2k - 2$ then the feasible pairs (j, m) which makes $\alpha_{k,j}$ undefined are $(j_0, k-1), (j_0+1, k-2) \dots, (j_0+k-1, 0)$ where $j_0 = m_0 - k + 1$. We can then write $f_{\tau^k}(t)$ of (9) as

$$f_{\tau^k}(t) = \sum_{j=0}^{m_0-k} \alpha_{k,j} a e^{-\beta_j a t} + \sum_{j=m_0-k+1}^{(m_0-1)/2} \left(\alpha_{k,j} a e^{-\beta_j a t} + \alpha_{k,m_0-j} a e^{-\beta_{m_0-j} a t} \right).$$

The first term is well defined and the second term can be redefined in the same way as the previous case.

Now consider the case when m_0 is an even number. The discussion is the same as the case when m_0 is odd. The only difference is that $\alpha_{k,m_0/2}$ has no zero factors in the denominator and is well defined. For example, if $0 \leq m_0 \leq k - 1$ then $f_{\tau^k}(t)$ equals

$$\sum_{j=0}^{(m_0-2)/2} \left(\alpha_{k,j} a e^{-\beta_j a t} + \alpha_{k,m_0-j} a e^{-\beta_{m_0-j} a t} \right) + \alpha_{k,m_0/2} a e^{-\beta_{m_0/2} a t} + \sum_{j=m_0+1}^{k-1} \alpha_{k,j} a e^{-\beta_j a t}.$$

The second and third terms are well defined and the first term can be redefined in the same way as in the case when m_0 is an odd number. $\square$

Proof of Lemma 3. For $t_k < t_B$ we have

$$P(\tau^k < t_k, \tau^B > t_B) = \sum_{i=k}^n P(\tau^k < t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B)$$

where $\tau^{n+1} = \infty$, and

$$E(1_{\{\tau^B > t_B\}} | \tau^i < t_B, \tau^{i+1} > t_B) = e^{b_1 \tau^1 + \dots + b_i \tau^i - (b_0 + \dots + b_i) t_B}.$$

Denote by

$$C_i = \frac{n! a^i}{(n-i)!} \prod_{m=1}^{i-1} (1 + m c) \text{ and } \tilde{w}_l = w_l - b_l. \quad (24)$$

We have

$$\begin{aligned} & P(\tau^k < t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B) \\ &= E_{\tau} \left[1_{\{\tau^1 < \dots < \tau^k < t_k\}} 1_{\{\tau^k < \dots < \tau^i < t_B\}} 1_{\{t_B < \tau^{i+1} < \dots < \tau^n\}} E(1_{\{\tau^B > t_B\}} | \tau) \right] \\ &= \int_0^{t_k} \int_{s_1}^{t_k} \dots \int_{s_{k-1}}^{t_k} \int_{s_k}^{t_B} \int_{s_{k+1}}^{t_B} \dots \int_{s_{i-1}}^{t_B} \int_{t_B}^{\infty} \int_{s_{i+1}}^{\infty} \dots \int_{s_{n-1}}^{\infty} \end{aligned}$$

$$\begin{aligned}
& n! C e^{-\sum_{l=1}^n w_l s_l} E(1_{\{\tau^B > t_B\}} | \tau^1 = s_1, \dots, \tau^n = s_n) ds_1 \dots ds_n \\
= & \int_0^{t_k} \int_{s_1}^{t_k} \dots \int_{s_{k-1}}^{t_k} \int_{s_k}^{t_B} \int_{s_{k+1}}^{t_B} \dots \int_{s_{i-1}}^{t_B} \int_{t_B}^{\infty} \int_{s_{i+1}}^{\infty} \dots \int_{s_{n-1}}^{\infty} \\
& n! C e^{-\sum_{l=1}^n w_l s_l} \cdot e^{\sum_{l=1}^i b_l s_l - (\sum_{l=0}^i b_l) t_B} ds_1 \dots ds_n \\
= & C_i e^{-(\sum_{l=0}^i b_l + \sum_{l=i+1}^n w_l) t_B} \\
& \cdot \int_0^{t_k} \int_{s_1}^{t_k} \dots \int_{s_{k-1}}^{t_k} \int_{s_k}^{t_B} \int_{s_{k+1}}^{t_B} \dots \int_{s_{i-1}}^{t_B} e^{-\sum_{l=1}^i \tilde{w}_l s_l} ds_1 \dots ds_i \\
= & C_i e^{-(\sum_{l=0}^i b_l + \sum_{l=i+1}^n w_l) t_B} \int_0^{t_k} \int_{s_1}^{t_k} \dots \int_{s_{k-1}}^{t_k} e^{-\sum_{l=1}^k \tilde{w}_l s_l} \\
& \cdot \sum_{j=0}^{i-k} \gamma_{i-k,j}(\tilde{w}_{k+1}, \dots, \tilde{w}_i) e^{-(\sum_{l=k+1+j}^i \tilde{w}_l) t_B - (\sum_{l=k+1}^{k+j} \tilde{w}_l) s_k} ds_1 \dots ds_k \\
= & \sum_{j=0}^{i-k} C_i \gamma_{i-k,j}(\tilde{w}_{k+1}, \dots, \tilde{w}_i) e^{-(\sum_{l=0}^{k+j} b_l + \sum_{l=k+j+1}^n w_l) t_B} A_k(t_k, \tilde{w}_1, \dots, \tilde{w}_{k-1}, \sum_{l=k}^{k+j} \tilde{w}_l)
\end{aligned}$$

Differentiating $P(\tau^k < t_k, \tau^B > t_B)$ with respect to t_k and t_B , and exchanging the order of summation (first i and j then i and j_1) give the joint density function f_{τ^k, τ^B} as follows:

$$f_{\tau^k, \tau^B}(t_k, t_B) = \sum_{j=k}^n \sum_{j_1=0}^{k-1} \alpha_{k,j,j_1} e^{-\beta_{j,j_1} t_B - \gamma_{j,j_1} t_k}$$

for $t_k < t_B$, where coefficients are given by $\beta_{j,j_1} = \sum_{l=0}^j b_l + \beta_j a$, $\gamma_{j,j_1} = \beta_{j_1} a - \beta_j a$, and

$$\alpha_{k,j,j_1} = \sum_{i=j}^n C_i \gamma_{i-k,j-k}(\tilde{w}_{k+1}, \dots, \tilde{w}_i) \gamma_{k-1,j_1}(\tilde{w}_1, \dots, \tilde{w}_{k-1}) \beta_{j,j_1} \quad (25)$$

and γ_{ij} , C_i , $\tilde{w}_l$ are given by (23) and (24). Similarly, for $t_k > t_B$ we have

$$P(\tau^k > t_k, \tau^B > t_B) = \sum_{i=0}^{k-1} P(\tau^k > t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B)$$

where $\tau^0 = 0$. Next we compute these probabilities

$$\begin{aligned}
& P(\tau^k > t_k, \tau^i < t_B, \tau^{i+1} > t_B, \tau^B > t_B) \\
= & E_{\tau} \left[1_{\{\tau^1 < \dots < \tau^i < t_B\}} 1_{\{t_B < \tau^{i+1} < \dots < \tau^{k-1} < \tau^k\}} 1_{\{t_k < \tau^k\}} 1_{\{\tau^k < \tau^{k+1} < \dots < \tau^n\}} E(1_{\{\tau^B > t_B\}} | \tau) \right]
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{t_B} \int_{s_1}^{t_B} \dots \int_{s_{i-1}}^{t_B} n! C e^{-(\sum_{l=1}^i \tilde{w}_l s_l) - (\sum_{l=0}^i b_l) t_B} ds_1 \dots ds_i \\
&\quad \cdot \int_{t_k}^{\infty} \left(\int_{t_B}^{s_k} \int_{s_{i+1}}^{s_k} \dots \int_{s_{k-2}}^{s_k} e^{-\sum_{l=i+1}^{k-1} w_l s_l} ds_{i+1} \dots ds_{k-1} \right) \\
&\quad \cdot \left(\int_{s_k}^{\infty} \int_{s_{k+1}}^{\infty} \dots \int_{s_{n-1}}^{\infty} e^{-\sum_{l=k+1}^n w_l s_l} ds_{k+1} \dots ds_n \right) e^{-w_k s_k} ds_k \\
&= C_k e^{-(\sum_{l=0}^i b_l) t_B} \sum_{j_1=0}^i \gamma_{i,j_1}(\tilde{w}_1, \dots, \tilde{w}_i) e^{-(\sum_{l=j_1+1}^i \tilde{w}_l) t_B} \\
&\quad \cdot \int_{t_k}^{\infty} \left(\sum_{j=0}^{k-1-i} \gamma_{k-1-i,j}(w_{i+1}, \dots, w_{k-1}) e^{-(\sum_{l=i+j+1}^{k-1} w_l) s_k - (\sum_{l=i+1}^{i+j} w_l) t_B} \right) \\
&\quad \cdot e^{-(\sum_{l=k}^n w_l) s_k} ds_k \\
&= \sum_{j=0}^{k-1-i} \sum_{j_1=0}^i \frac{C_k \gamma_{k-1-i,j}(w_{i+1}, \dots, w_{k-1}) \gamma_{i,j_1}(\tilde{w}_1, \dots, \tilde{w}_i)}{(\sum_{l=i+j+1}^n w_l)} \\
&\quad \cdot e^{-(\sum_{l=0}^{j_1} b_l + \sum_{l=j_1+1}^{i+j} w_l) t_B - (\sum_{l=i+j+1}^n w_l) t_k}
\end{aligned}$$

Differentiating $P(\tau^k > t_k, \tau^B > t_B)$ with respect to t_k and t_B and exchanging the order of summation (first i and j then i and j_1) give the joint density function f as follows:

$$f_{\tau^k, \tau^B}(t_k, t_B) = \sum_{j=0}^{k-1} \sum_{j_1=0}^j \tilde{\alpha}_{k,j,j_1} e^{-\tilde{\beta}_{j,j_1} t_B - \tilde{\gamma}_{j,j_1} t_k}$$

for $t_k > t_B$, where coefficients are given by $\tilde{\beta}_{j,j_1} = \sum_{l=0}^{j_1} b_l + \beta_{j_1} a - \beta_j a$, $\tilde{\gamma}_{j,j_1} = \beta_j a$, and

$$\tilde{\alpha}_{k,j,j_1} = \sum_{i=j_1}^j C_k \gamma_{k-1-i,j-i}(w_{i+1}, \dots, w_{k-1}) \gamma_{i,j_1}(\tilde{w}_1, \dots, \tilde{w}_i) \tilde{\beta}_{j,j_1} \quad (26)$$

□

Proof of Lemma 4. Given a complete sample path $\{X(t) : t \geq 0\}$ the function $X(t)$ is known and deterministic. Then for $t_1 < t_2 < \dots < t_n$

$$c(t_1, \dots, t_n) = X(t_1) X(t_2) \dots X(t_n) \prod_{m=1}^{n-1} (1 + mc)$$

and $w_l(u) = c_l X(u)$ where $c_l = (1 + (-n + 2l - 1)c)$ for $i = 1, \dots, k$. We can then show in the same method as in the proofs of Lemmas 1 and 2 that

$$\begin{aligned} & \int_t^\infty \int_{t_{k+1}}^\infty \dots \int_{t_{n-1}}^\infty e^{-\sum_{l=k+1}^n (c_l \int_0^{t_l} X(u) du)} \cdot X(t_{k+1}) \dots X(t_n) dt_{k+1} \dots dt_n \\ &= \frac{1}{\prod_{m=k+1}^n (\sum_{l=m}^n c_l)} e^{-(\sum_{l=k+1}^n c_l) \int_0^t X(u) du} \end{aligned}$$

and

$$\begin{aligned} & \int_0^t \int_{t_1}^t \dots \int_{t_{k-1}}^t e^{-\sum_{l=1}^k c_l \int_0^{t_l} X(u) du} \cdot X(t_1) \dots X(t_k) dt_1 \dots dt_k \\ &= \sum_{j=0}^k \gamma_{k,j}(c_1, \dots, c_k) e^{-(\sum_{l=j+1}^k c_l) \int_0^t X(u) du}. \end{aligned}$$

□

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