On Majorana Representations of A_6 and A_7

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Abstract

The Majorana representations of groups were introduced in [Iv09] by axiomatising some properties of the $2A$ -axial vectors of the 196 884-dimensional Monster algebra, inspired by the sensational classification of such representations for the dihedral groups achieved by S. Sakuma in [Sak07]. This classification took place in the heart of the theory of Vertex Operator Algebras and expanded earlier results by M. Miyamoto [Miy03]. Every subgroup G of the Monster which is generated by its intersection with the conjugacy class of $2A$ -involutions possesses a Majorana representation obtained by restricting to G the action of the Monster on its algebra. This representation of G is said to be *based on an embedding of G in the Monster*. So far the Majorana representations have been classified for the groups S_4 [IPSS10], A_5 [ISe10], and $L_3(2)$ [ISh10]. All these representations are based on embeddings in the Monster; their dimensions and shapes are given in the following table.

"What is our life? A game!" (A.S. Pushkin, "The Queen of Spades")

Shape	$(2A, 3A)$	$(2A, 3C)$	$(2B, 3A)$	$(2B, 3C)$
S_4	13	9	13	6
A_5	26	20	—	—
$L_3(2)$	49	21	—	—

In the present note the classification is expanded to the groups A_6 and A_7 (subject to invariance of the $3A$ -axial vectors and the absence of $3C$ -subalgebras).

1 Introduction

A tuple

$$\mathcal{R} = (G, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)$$

is said to be a *Majorana representation* (of G) if the following conditions hold: G is a finite group; T is a set of involutions (elements of order 2) in G which generates G and stable under conjugation by elements of G (this means that T is a generating union of conjugacy classes of involutions in G); V is a real vector space equipped with an inner product $(\cdot, \cdot)$ and with a commutative non-associative algebra product $\cdot$, which associate with each other and satisfy the *Norton inequality*: $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$ for all $u, v \in V$;

$$\varphi : G \rightarrow GL(V)$$

is a faithful homomorphism, whose image preserves $(\cdot, \cdot)$ and $\cdot$; and ψ is a rule which assigns to every $t \in T$ a vector $a_t = \psi(t)$ which is a *Majorana axis* in $(V, (\cdot, \cdot), \cdot)$ (cf. the definition in the next paragraph) and such that the Majorana involution, associated with a_t coincides with $\varphi(t)$. It is further assumed that V is generated by the elements a_t taken for all $t \in T$, and $\dim(V)$ is said to be the *dimension* of $\mathcal{R}$. It is assumed that φ and ψ commute in the sense that

$$a_{g^{-1}tg} = (a_t)^{\varphi(g)}$$

for every $g \in G$. For a subset X of vectors in V we denote the linear span of X in V and the algebra closure of X in $(V, \cdot)$ by

$$\langle X \rangle \text{ and } \ll X \gg$$

respectively. Thus $\ll X \gg$ is the smallest subspace Y in V containing X such that $y_1 \cdot y_2 \in Y$ whenever $y_1, y_2 \in Y$.¹

By the definition in [Iv09] a Majorana axis a in $(V, (\cdot, \cdot), \cdot)$ is an idempotent of length 1, whose adjoint operator

$$\text{ad}_a : v \mapsto a \cdot v$$

is semi-simple with spectrum $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$ (when talking about eigenvectors of a one really means eigenvectors of ad_a). The following conditions concerning the eigenspaces are imposed. The 1-eigenvectors of a are precisely the scalar multiples of a . The Majorana involution $\tau(a)$ associated with a is the linear transformation of V which negates every $\frac{1}{32}$ -eigenvector and centralizes the other eigenvectors. By the above definition

¹It was rightfully pointed out by the referee that only a centerless group might possess a Majorana representation.

the Majorana involution is an automorphism of $(V, (,), \cdot)$. This condition is equivalent to the *fusion rules* involving $\frac{1}{32}$ -eigenvectors: if v and u are $\frac{1}{32}$ -eigenvectors of a , if x and y are λ - and μ -eigenvectors for $\lambda, \mu \in \{1, 0, \frac{1}{4}\}$ then $v \cdot x$ is a $\frac{1}{32}$ -eigenvector, and both $v \cdot u$ and $x \cdot y$ project to zero in the $\frac{1}{32}$ -eigenspace. The remaining fusion rules are as follows: if α_1 and α_2 are 0-eigenvectors of a , and β_1 and β_2 are $\frac{1}{4}$ -eigenvectors of a , then

$$\alpha_1 \cdot \alpha_2, \quad \beta_1 \cdot \beta_2 - (\beta_1 \cdot \beta_2, a) a \quad \text{and} \quad \alpha_1 \cdot \beta_1$$

are λ -eigenvectors of a , where $\lambda = 0, 0$ and $\frac{1}{4}$, respectively.

Sakuma's theorem [Sak07] together with Norton's explicit description of the subalgebras in the Monster algebra generated by pairs of transposition axes [N96], implies the following proposition, which will now be installed in axiomatics of the Majorana theory.

Proposition 1 [Sak07], [N96] *Let $\mathcal{R} = (G, T, V, (,), \cdot, \varphi, \psi)$ be a Majorana representation. For two distinct involutions t_0 and t_1 in T put $a_0 = \psi(t_0)$, $a_1 = \psi(t_1)$, $\tau_0 = \varphi(t_0)$, $\tau_1 = \varphi(t_1)$, $\rho = t_0 t_1$. Let D be the dihedral subgroup in $GL(V)$ generated by τ_0 and τ_1 , and let $|D| = 2N$. Then the subalgebra $Y = \ll a_0, a_1 \gg$ is isomorphic to one of the eight Norton–Sakuma algebras in Table 1 (more specifically to an algebra of type NX , where N is as above and $X \in \{A, B, C\}$). For an integer i and $\varepsilon \in \{0, 1\}$ the vector $a_{2i+\varepsilon}$ is the image of a_ε under the i -th power of ρ (so that $\tau(a_{2i+\varepsilon}) = \rho^{-i} \tau_\varepsilon \rho^i$), and the remaining vectors in the basis of Y given in the second column in Table 1 are centralized by D . The kernel of the action of D on Y coincides with the centre of D .*

For the case of algebras generated by two Majorana axes it was shown in [IPSS10] (cf. the beginning of Section 2) that the Norton inequality is a consequence of the other Majorana axioms.

The *shape* of a Majorana representation $\mathcal{R}$ is a rule which specifies the type of the Norton–Sakuma subalgebra $\ll \psi(t_0), \psi(t_1) \gg$ for every pair t_0, t_1 of involutions in T . The rule must be stable under conjugation by the elements of G and must respect the embeddings of the algebras:

$$2A \hookrightarrow 4B, \quad 2A \hookrightarrow 6A, \quad 2B \hookrightarrow 4A, \quad 3A \hookrightarrow 6A.$$

The $2A$ -subalgebras in the Monster possess an important property which we axiomatise:

(2A) the following conditions hold (where $t_0, t_1, t_2 \in T$ and $a_i = \psi(t_i)$ for $0 \leq i \leq 2$):

- (a) if $t_0 t_1 t_2 = 1$, then $\ll a_0, a_1 \gg$ is of type $2A$ and $a_2 = a_\rho := a_0 + a_1 - 8a_0 \cdot a_1$;
- (b) if $\ll a_0, a_1 \gg$ is of type $2A, 4B$ or $6A$, then $t_0 t_1$, $(t_0 t_1)^2$ or $(t_0 t_1)^3$ belongs to T , and $\psi(t_0 t_1)$, $\psi((t_0 t_1)^2)$ or $\psi((t_0 t_1)^3)$ coincides with a_ρ , a_{ρ^2} or a_{ρ^3} .

Table 1

Type	Basis	Products and angles
2A	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho), \quad a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$ $(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{2^3}$
2B	a_0, a_1	$a_0 \cdot a_1 = 0, \quad (a_0, a_1) = 0$
3A	a_{-1}, a_0, a_1, u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}}u_\rho$ $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5}u_\rho$ $u_\rho \cdot u_\rho = u_\rho$ $(a_0, a_1) = \frac{13}{2^8}, \quad (a_0, u_\rho) = \frac{1}{2^2}, \quad (u_\rho, u_\rho) = \frac{2^3}{5}$
3C	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1}), \quad (a_0, a_1) = \frac{1}{2^6}$
4A	$a_{-1}, a_0, a_1, a_2, v_\rho$	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_2 + a_{-1} - 3v_\rho)$ $a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_\rho)$ $v_\rho \cdot v_\rho = v_\rho, \quad a_0 \cdot a_2 = 0$ $(a_0, a_1) = \frac{1}{2^5}, \quad (a_0, a_2) = 0, \quad (a_0, v_\rho) = \frac{3}{2^3}, \quad (v_\rho, v_\rho) = 2$
4B	$a_{-1}, a_0, a_1, a_2, a_{\rho^2}$	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$ $a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$ $(a_0, a_1) = \frac{1}{2^6}, \quad (a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{2^3}$
5A	$a_{-2}, a_{-1}, a_0, a_1, a_2, w_\rho$	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5}w_\rho$ $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $(a_0, a_1) = \frac{3}{2^7}, \quad (a_0, w_\rho) = 0, \quad (w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
6A	$a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_{\rho^3}, u_{\rho^2}$	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}}u_{\rho^2}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}}u_{\rho^2}$ $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5}u_{\rho^2}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3}), \quad a_{\rho^3} \cdot u_{\rho^2} = 0, \quad (a_{\rho^3}, u_{\rho^2}) = 0$ $(a_0, a_1) = \frac{5}{2^8}, \quad (a_0, a_2) = \frac{13}{2^8}, \quad (a_0, a_3) = \frac{1}{2^3}$

In Section 7 of [ISe10] a Majorana representation of A_5 was constructed, in which $\ll a_t, a_s \gg$ is a $2B$ -algebra whenever t and s are commuting involutions. Since A_5 contains a single class of involutions the product ts must be in T , and hence the condition (2A) does not hold for that representation.

The vectors u_ρ , v_ρ and $\pm w_\rho$ in the $3A$ -, $4A$ - and $5A$ -subalgebras (cf. Table 1) in the Monster case are called $3A$ -, $4A$ - and $5A$ -axial vectors, respectively. We apply this terminology to an arbitrary Majorana representation. In the case of the Monster group M , the axial vector associated with ρ can be specified inside the centralizer of $C_M(\rho)$ in the 196 884-dimensional vector space. In the present setting the axial vectors are defined through the subalgebras generated by pairs of Majorana generators. It is natural to assume that every axis is independent of the choice of the subalgebra involved in its definitions. In the groups we have considered so far (S_4 , A_5 and $L_3(2)$), every cyclic subgroup of order n is contained in at most one dihedral subgroup of order $2n$. The situation in A_6 is different since it contains a pair of D_6 -subgroups intersecting in a cyclic subgroup of order 3. This forces us to axiomatize yet another property of the Majorana representation of the Monster (we don't know whether or not this axiom is independent of the previous Majorana axioms).

(3A) the following condition holds (where $t_0, t_1, t_2, t_3 \in T$,

$a_i = \psi(t_i)$ for $0 \leq i \leq 3$): if

(a) $\langle t_0, t_1 \rangle \cong \langle t_2, t_3 \rangle \cong D_6$;

(b) $t_0 t_1 = t_2 t_3$;

(c) both $\ll a_0, a_1 \gg$ and $\ll a_2, a_3 \gg$ have type $3A$;

then the $3A$ -axial vectors in the subalgebras in (c) are equal.

If G is A_6 or A_7 then every cyclic subgroup of order 5 of G is contained in a unique dihedral subgroup of order 10, so the uniqueness condition for $5A$ is not necessary in these cases. In fact, the uniqueness property for $2A$ -generated D_{10} -subgroups holds in the whole of the Monster, so we might never need to axiomatise it. On the other hand, it is doubtful that the Majorana representations of A_8 can be classified without a uniqueness assumption of the $4A$ -axial vectors.

Let $\mathcal{R} = (G, T, V, (,), \cdot, \varphi, \psi)$ be a Majorana representation of a group G and let

$$\zeta : g \mapsto g^\vartheta$$

be an automorphism of G preserving T . Then $\mathcal{R}$ is said to be ζ -stable if the permutation which maps $\psi(t)$ onto $\psi(t^\vartheta)$ for every $t \in T$ extends to an automorphism of

$(V, (\ , \), \cdot)$. S -stability (where S is a group of automorphisms of G) has the obvious meaning. By definition every Majorana representation of G is stable under the inner automorphisms of G .

It is implicit in Table 1 that every element of order 5 in the Monster which is inverted by a $2A$ -involution has type $5A$. All simple subgroups in the Monster containing $5A$ -elements have been classified by Simon Norton. The classification proof remains unpublished but the list of subgroups can be found in [N98]. It can be seen from this list that the Monster contains a single class of A_6 -subgroups and a single class of A_7 subgroups which are generated by $2A$ involutions. These subgroups are 6- and 5-point stabilizers in the ‘standard’ A_{12} -subgroup in the Monster, which is the centralizer of an A_5 -subgroup of type $(2A, 3A, 5A)$. All 3-elements in the A_6 and A_7 -subgroups in question are of type $3A$. Finally, each of these two subgroups is *fully normalized* in the Monster in the sense that every automorphism is induced by a suitable element from the normalizer. This property can be seen from the structure of the Monstralizers quoted in the proof of Lemma 5.1.

Theorem 1 *Let G be isomorphic to the alternating group of degree 6 or 7, and let $\mathcal{R} = (G, T, V, (\ , \), \cdot, \varphi, \psi)$ be a Majorana representation of G satisfying the conditions (2A) and (3A). Suppose further that whenever t_0, t_1 are involutions in T which generate a D_6 -subgroup, the subalgebra $\ll \psi(t_0), \psi(t_1) \gg$ has type $3A$ (rather than $3C$). Then*

- (i) $\mathcal{R}$ is uniquely determined by G ;
- (ii) $\mathcal{R}$ is based on an embedding of G in the Monster and it is ζ -stable for every $\zeta \in \text{Aut}(G)$;
- (iii) if $G \cong A_6$ then $\dim(V)$ is either 75 or 76;
- (iv) if $G \cong A_7$ then $\dim(V)$ is either 196 or 197.

It follows from the properties of the Monster stated in the paragraph before the theorem that the second assertion would follow from the first.²

Having been challenged by the referee’s remark on an earlier version of the paper I was able to prove the following.

Proposition 1.1 [Iv11] *Suppose there exists a Majorana representation $\mathcal{R} = (A_6, T, V, (\ , \), \cdot, \varphi, \psi)$ such that the subalgebra $\ll \psi(t_0), \psi(t_1) \gg$ has type $3C$ for at least one pair t_0, t_1 of involutions in T which generate a D_6 -subgroup. Then*

²Recently the representations of Theorem 1 have been explicitly constructed by Ákos Seress (private communication). Their dimensions turned out to be 76 and 196, respectively.

- (i) $\ll \psi(t), \psi(s) \gg$ has type $3C$ for all pairs t, s of involutions in T which generate a D_6 -subgroup;
- (ii) $\mathcal{R}$ is 2-closed and 70-dimensional;
- (iii) $\mathcal{R}$ is determined uniquely up to isomorphism. □

Corollary 1.2 *There exist no Majorana representations of A_7 for which the subalgebra $\ll \psi(t_0), \psi(t_1) \gg$ has type $3C$ for at least one pair t_0, t_1 of involutions generating a D_6 -subgroup.*

Proof. If $t_0 t_1$ is a 3-cycle, then $t_0 t_1$ is contained in a D_{12} -subgroup and hence $\ll \psi(t_0), \psi(t_1) \gg$ must be of type $3A$, since the $3C$ -algebra is not contained in the $6A$ -type algebra. If $\ll \psi(t_0), \psi(t_1) \gg$ has type $3C$ when $t_0 t_1$ is the product of two 3-cycles, then either the preceding argument applies or the restriction of the Majorana representation of A_7 to an A_6 -subgroup involves both $3A$ - and $3C$ -types, contrary to Proposition 1.1 (i). □

In the proof of Theorem 1 we make use of the fact that the restriction of $\mathcal{R}$ to a subgroup S_4 , A_5 or $L_3(2)$ in G is an explicitly known Majorana representation of shape $(2A, 3A)$ explicitly described in [IPSS10], [ISe10] or [ISh10], respectively. This fact enables us to calculate many pairwise products in V . The remaining ones are recovered by a formulae introduced in [IPSS10] under the name of the *resurrection principle*. This principle is an immediate consequence of the fusion rule and we reproduce it below for the sake of completeness (compare Lemma 1.7 in [IPSS10]).

Lemma 1.3 *Let $\mathcal{R} = (G, T, V, (,), \cdot, \varphi, \psi)$ be a Majorana representation, let a be a Majorana axis in V , and let X be an a -stable subspace of V in the sense that $a \cdot x \in X$ for every $x \in X$. Suppose that $s \in V$ and that*

$$\alpha_s = s + x_0 \text{ and } \beta_s = s + x_{\frac{1}{4}}$$

are, respectively, 0- and $\frac{1}{2^2}$ -eigenvectors of a for some $x_0, x_{\frac{1}{4}} \in X$. Then

$$s = - \left[4a \cdot (x_0 - x_{\frac{1}{4}}) + x_{\frac{1}{4}} \right],$$

in particular $s \in X$. □

The results in this note would not have been achieved without essential computational and conceptual support provided by Igor Faradjev, Simon Norton, Dima Pasechnik, Ákos Seress, and Sergey Shpectorov.

2 2-closure

Let $\mathcal{R} = (G, T, V, (\ , \), \cdot, \varphi, \psi)$ be a Majorana representation. As above for $t \in T$ we denote by a_t the corresponding Majorana axis $\psi(t)$. Put

$$A = \{a_t \mid t \in T\} \text{ and } A^2 = \langle a_t \cdot a_s \mid t, s \in T \rangle$$

(note that A^2 contains A , since the Majorana axes are idempotents). By definition $V = \ll A \gg$. A Majorana representation said to be *2-closed* if $V = \langle A^2 \rangle$. It is clear that $\mathcal{R}$ is 2-closed if and only if the algebra product $\cdot$ is closed on the linear span $\langle A^2 \rangle$ of A^2 .

As a consequence of the main results of [IPSS10], [ISe10], [ISh10] we have the following result.

Lemma 2.1 *Let H be S_4 , A_5 or $L_3(2)$, and let $\mathcal{R}$ be a Majorana representation of H . Then either*

- (i) $\mathcal{R}$ is 2-closed, or
- (ii) $H \cong S_4$ and $\mathcal{R}$ is of shape $(2B, 3A)$ and the algebra product is closed on the linear span of the set

$$\{a_t \cdot a_s \cdot a_r \mid t, s, r \in T\}.$$

□

From now on we assume that G is the alternating group of degree 6 or 7 and that $\mathcal{R} = (G, T, V, (\ , \), \cdot, \varphi, \psi)$ is a Majorana representation of G which satisfies the hypothesis of Theorem 1.

All the involutions in G are conjugate, which implies that T is the set of all involutions in G , so all the involutions in G are of type $2A$. We are going to show that $\ll A^2 \gg$ contains a spanning set whose vectors are indexed by the subgroups of order 2, 3 and 5 in G . By definition $\ll A^2 \gg$ is linearly spanned by A together with the subalgebras $Y = \ll a_t, a_s \gg$ taken for all $a_t, a_s \in A$. By Proposition 1, unless $a_t = a_s$, the subalgebra Y is isomorphic to a Norton–Sakuma algebra of type NX in Table 1, where N is the order of ts . We analyse the possibilities for Y , depending on N .

If $N = 2$ then $ts \in T$ since T contains all the involutions of G and by (2A) Y must be of type $2A$ with

$$Y = \langle a_t, a_s, a_{ts} \rangle \subseteq \langle A \rangle.$$

If $N = 3$ then by the hypothesis of Theorem 1 Y is of type 3A. By Proposition 1 and since φ and ψ commute, we have

$$Y = \langle a_t, a_s = a_{hth^{-1}}, a_{h^{-1}th}, u_h(t) \rangle,$$

where $h = ts$ and

$$u_h(t) = \frac{2^{11}}{3^3 \cdot 5} \left(\frac{1}{2^5} (2a_t + 2a_{hth^{-1}} + a_{h^{-1}th}) - a_t \cdot a_{hth^{-1}} \right)$$

is the unique idempotent in Y of squared norm $\frac{8}{5}$.

Since Y is stable under the φ -image of $D = \langle t, s \rangle \cong D_6$ and since $\varphi(t)$ maps $a_{hth^{-1}}$ onto $a_{h^{-1}th}$ centralizing a_t , the vector $u_{h^{-1}}(t)$ defined via the pair $(a_t, a_{h^{-1}th})$ is equal to $u_h(t)$ by the last assertion in Proposition 1. By (3A) the vector $u_h(t)$ does not depend on the particular choice of the dihedral group $D = \langle h, t \rangle$ of order 6 containing $\langle h \rangle$ and can be denoted simply by u_h (by the above $u_h = u_{h^{-1}}$). Finally, every subgroup of order 3 in G is inverted by an involution and hence it is contained in a D_6 -subgroup. Therefore with every subgroup of order 3 in G we have associated a vector u_h (where h is a generator of the subgroup), so that the linear span of the union of these vectors with the set A contains the linear span of the 2-generated subalgebras Y of type 3A.

If $N = 4$ then Y must be of type 4B, since the 4A-algebra contains a 2B-subalgebra, which is not present by the $N = 2$ case. By (2A) we have

$$Y = \langle a_t, a_s, a_{tst}, a_{sts}, a_{(ts)^2} \rangle \subseteq \langle A \rangle.$$

If $N = 5$ then Y is of type 5A and with $f = ts$ we have

$$Y = \langle a_t, a_s = a_{f^2tf^3}, a_{f^3tf^2}, a_{ftf^{-1}}, a_{f^{-1}tf}, w_f(t) \rangle,$$

where

$$w_f(t) = a_t \cdot a_{f^2tf^3} - \frac{1}{2^7} (3a_t + 3a_{f^2tf^3} - a_{ftf^{-1}} - a_{f^{-1}tf} - a_{f^3tf^2}).$$

If F is the automorphism group of the algebra structure on Y , preserving $\{a_{f^i t f^{5-i}} \mid 1 \leq i \leq 4\}$, then F is easily seen to be F_{20} , the Frobenius group of order 20, and, up to rescaling, $w_f(t)$ is the unique vector in Y which is centralized by the elements of the D_{10} -subgroup in F and negated by the remaining elements of F .

The structure of the 5A-algebra and its invariance under $\langle t, s \rangle \cong D_{10}$ show that

$$w_f(t) = -w_{f^2}(t) = -w_{f^3}(t) = w_{f^{-1}}(t).$$

It is clear, that $w_f(t)$ does not depend of the particular choice of the involution t inside the D_{10} -subgroup containing f . Since

$$N_{A_6}(Z_5) \cong D_{10}, \quad N_{A_7}(Z_5) \cong F_{20},$$

this D_{10} -subgroup is uniquely determined by f and (without the (5A)-condition) with every subgroup of order 5 in G we can associate a 1-dimensional subspace of $\ll A^2 \gg$ spanned by w_f where f is a generator of the subgroup.

If $N = 6$ then Y is of type 6A and with $g = ts$ and by (2A) we have

$$Y = \langle a_t, a_s, a_{gtg^{-1}}, a_{g^{-1}tg}, a_{gsg^{-1}}, a_{g^{-1}sg}, a_{(st)^3}, u_{g^2} \rangle.$$

Since $\ll a_t, a_{gtg^{-1}} \gg$ is a subalgebra of type 3A which contains u_{g^2} (since $g^2 = tg^{-1}tg$), the algebras of type 6A are already generated as vector spaces by their 2A-axials and the vectors produced in the 3A-subalgebras.

For $i = 2, 3$ or 5 let $G^{(i)}$ denote a set of elements of order i in G which contains one representative from every subgroup of order i in G ; if $x, y \in G^{(i)}$ and the subgroups $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G we choose x and y to be G -conjugate. Clearly $G^{(2)}$ is the set T of involutions in G . The above considerations can be summarised in the following lemma.

Lemma 2.2 *Let $A = \{a_t \mid t \in G^{(2)}\}$, $U = \{u_h \mid h \in G^{(3)}\}$ and $W = \{w_f \mid f \in G^{(5)}\}$. Then $\langle A^2 \rangle = \langle A \cup U \cup W \rangle$. $\square$*

For a subgroup H of G and $i \in \{2, 3, 5\}$ put $H^{(i)} = H \cap G^{(i)}$ and define

$$A(H) = \{a_t \mid t \in H^{(2)}\}, \quad U(H) = \{u_h \mid h \in H^{(3)}\}, \quad W(H) = \{w_f \mid f \in H^{(5)}\},$$

$$V(H) = A(H) \cup U(H) \cup W(H).$$

Lemma 2.3 *Let H be a subgroup of G isomorphic to D_{12} , A_4 , S_4 , A_5 or $L_3(2)$. Then $\ll V(H) \gg = \langle V(H) \rangle$.*

Proof. If $H \cong D_{12}$ then Proposition 1 is applicable. If H is S_4 , A_5 or $L_3(2)$ then the restriction of $\mathcal{R}$ to H is a Majorana representation of H of shape (2A, 3A) and Lemma 2.1 is applicable. Every A_4 -subgroup of G is contained in an S_4 -subgroup, and by Lemma 4.16 in [IPSS10] in the Majorana representation of S_4 the algebra product is closed on $V(A_4)$. This provides the required reduction. $\square$

Lemma 2.4 *Let F_{21} be a Frobenius subgroup of order 21 in A_7 . Then F_{21} is contained in a subgroup $H \cong L_3(2)$ of A_7 and $\ll V(F_{21}) \gg \subseteq \langle V(H) \rangle$.*

Proof. Since F is the normalizer of a Sylow 7-subgroup in A_7 the inclusion of subgroups follows. By Lemma 2.3 the algebra product is closed on $V(H)$, hence we also have the required inclusion of subspaces. $\square$

Note that the explicit product formulas on $V(H)$ in the above lemma can be read from Table 1, [IPSS10], [ISe10], [ISh10]. Of course the $L_3(2)$ -subgroup appears only when G is A_7 .

As the crucial step in our treatment we show that the $5A$ -axial vectors in $\langle A^2 \rangle$ together add at most one dimension to the subspace $\langle A \cup U \rangle$ spanned by the Majorana and $3A$ -axial vectors. This conclusion is a consequence of *Norton's relation* stated on p. 300 of [N96].

Lemma 2.5 *The codimension of $\langle A \cup U \rangle$ in $\langle A \cup U \cup W \rangle$ is at most 1.*

Proof. A fundamental property of the Majorana representation of $H \cong A_5$ of the shape $(2A, 3A)$ is that $\langle A(H) \cup U(H) \rangle$ has codimension 1 in $V(H)$ (cf. p. 300 in [N96] and Lemma 4.4 in [ISe10]). Thus, in order to establish the assertion, it suffices to prove the connectivity of the graph $\Delta(G)$ on the set of order 5 subgroups in G , where two such subgroups are adjacent if they are contained in a common A_5 -subgroup. It is clear that the subgroups in a connected component of $\Delta(G)$ generate a nontrivial normal subgroup of G . Since both A_6 and A_7 are simple, the connectivity follows. $\square$

Lemma 2.6 *The Majorana representation $\mathcal{R}$ considered is 2-closed if and only if*

$$\ll A \cup U \gg \subseteq \langle A \cup U \cup W \rangle.$$

Proof. The ‘only if’ assertion is obvious. Suppose that the inclusion holds. In order to prove the 2-closure, in view of Lemma 2.2 it is sufficient to show that

$$x \cdot w_f \in \langle A \cup U \cup W \rangle$$

for every $f \in G^{(5)}$ and $x \in A \cup U$. Suppose that $x = a_t$ for some $t \in G^{(2)}$ (the case when $x = u_h$ can be treated similarly). Let g be an element of order 5 in G such that t and g are contained in a common A_5 -subgroup H . Then

$$a_t \cdot w_g \in A(H) \cup U(H) \cup W(H)$$

by Lemma 2.3, while by Lemma 2.5 $w_f = w_g + y$ for some $y \in \langle A \cup U \rangle$. Now the assumed inclusion gives the result. $\square$

We are going to prove the following enhanced version of the 2-closure condition stated in the above lemma.

Proposition 2.7 *The following assertions hold:*

- (i) *if $t, s \in G^{(2)}$ then $a_t \cdot a_s \in \langle A \cup U \cup W \rangle$;*
- (ii) *if $t \in G^{(2)}$ and $h \in G^{(3)}$ then $a_t \cdot u_h \in \langle A \cup U \cup W \rangle$;*
- (iii) *if $h, k \in G^{(3)}$ then $u_h \cdot u_k \in \langle A \cup U \cup W \rangle$;*
- (iv) *in each of the above cases the product is uniquely determined by the factors.*

Proof. The assertion (i) is immediate from Lemma 2.2. Roughly speaking (ii) hold because neither A_6 nor A_7 is generated by an element of order 2 together with an element of order 3. We start with the case $G \cong A_6$. For $t \in G^{(2)}$ and $h \in G^{(3)}$ let $o(th)$ denote the order of the group product of t and h and let $\langle t, h \rangle$ denote the isomorphism type of the subgroup in G generated by t and h . Then it is an elementary exercise to check that the possibilities are as described in (the first three columns of) the following table:

#	$o(th)$	$\langle t, h \rangle$	(a_t, u_h)
1	2	D_6	$\frac{1}{4}$
2	3	A_4	$\frac{1}{9}$
3	4	S_4	$\frac{1}{36}$
4	5	A_5	$\frac{1}{18}$

Thus in the A_6 -case assertion (ii) follows from Lemma 2.3. In the situation corresponding to the first row t normalizes $\langle h \rangle$ and thus $a_t \cdot u_h$ is contained in the $3A$ -subalgebra $\ll a_t, a_{h^{-1}th} \gg$. The entries in the last column of the above table can be deduced from [N96] Table 3 by rescaling or read from [IPSS10] and [ISe10].

Let us turn to the case $G \cong A_7$. If h is a 3-cycle then either t and h are contained in a common A_6 -subgroup and the assertion has already been established, or t and h have disjoint supports in the permutation action on seven points. In that case they are simultaneously inverted by an involution, so they are contained in a D_{12} -subgroup and again Lemma 2.3 is applicable (in this case a_t and u_h are perpendicular). We

may therefore assume that h is a product of two 3-cycles. It can be shown that under conjugation by S_7 there is a single orbit on pairs (t, h) with $o(th) = 7$. On the other hand, such a pair can be found inside an $L_3(2)$ -subgroup (it is a well known fact that A_7 is not a $(2, 3, 7)$ -group). Thus the table describing the possibilities for A_7 is the one for A_6 extended by the following two rows:

#	$o(th)$	$\langle t, h \rangle$	(a_t, u_h)
5	6	Z_6	0
6	7	$L_3(2)$	$\frac{1}{24}$

The rightmost entry of the last row can be deduced from [N96] Table 3 by rescaling or read from [ISh10]. Again Lemma 2.3 is applicable which completes the proof of (ii).

The proof of (iii) will be accomplished in Section 4 after some detailed analysis of $3A$ -algebras in the next section, but first we state explicitly what we have achieved so far.

Corollary 2.8 *The subspace $\langle A \cup U \cup W \rangle$ is Majorana stable in the sense that for every $t \in T$ it contains the product $a_t \cdot v$ whenever v taken from this subspace.*

Proof. The assertion is implied by Proposition 2.7 (i) and (ii) together with Lemma 2.6. $\square$

3 Calculating with $3A$ -subalgebras

We start with the following lemma, which can justified either by direct calculations with the formulas in Table 1 or by referring to Table 4 in [IPSS10].

Lemma 3.1 *Let $t \in T$, $h \in G^{(3)}$ and suppose that t inverts h , that is $tht = h^{-1}$. Then the vectors*

$$\alpha_h^{(t)} = u_h - \frac{2 \cdot 5}{3^3} a_t + \frac{2^5}{3^3} (a_{hth^{-1}} + a_{h^{-1}th})$$

and

$$\beta_h^{(t)} = u_h - \frac{2^3}{3^2 \cdot 5} a_t - \frac{2^5}{3^2 \cdot 5} (a_{hth^{-1}} + a_{h^{-1}th})$$

are, respectively, 0- and $\frac{1}{4}$ -eigenvectors of a_t in the algebra $\ll a_t, u_h \gg$ of type $3A$. $\square$

Proposition 3.2 *Let $t \in T$, and let $h, k \in G^{(3)}$. Suppose that h and k generate distinct subgroups in G and that t inverts both h and k . Then the following assertions hold:*

- (i) $(u_h, u_k) = \frac{2^4}{3^{5.5}}(5 - 2^3 \cdot 3^2 \cdot p + 2^7 \cdot q)$, where $p = (u_h, a_{ktk^{-1}})$, $q = (a_{hth^{-1}}, a_{ktk^{-1}}) + (a_{hth^{-1}}, a_{k^{-1}tk})$;
- (ii) $u_h \cdot u_k = -4a_t \cdot (\alpha_h^{(t)} \cdot \alpha_k^{(t)} - u_h \cdot u_k - \alpha_h^{(t)} \cdot \beta_k^{(t)} + u_h \cdot u_k) + \alpha_h^{(t)} \cdot \beta_k^{(t)} - u_h \cdot u_k$, where $\alpha_h^{(t)}$, $\alpha_k^{(t)}$, $\beta_h^{(t)}$ and $\beta_k^{(t)}$ are defined as in Lemma 3.1;
- (iii) if the subspace $\langle A \cup U \cup W \rangle$ is Majorana stable then it contains the product $u_h \cdot u_k$.

Proof. Since the inner and algebra products associate by the Majorana representation axiomatics, eigenvectors with distinct eigenvalues are perpendicular, and hence $(\alpha_h^{(t)}, \beta_k^{(t)}) = 0$. Expanding the latter equality in terms of the expressions for $\alpha_h^{(t)}$ and $\beta_k^{(t)}$ given in Lemma 3.1, we obtain (i). Note that the equality $(u_h, a_{ktk^{-1}}) = (u_k, a_{hth^{-1}})$ also follows from the associativity between the inner and algebra products. The assertion (ii) is an immediate consequences of the resurrection principle Lemma 1.3 for

$$s = u_h \cdot u_k, \quad \alpha_s^{(t)} = \alpha_h^{(t)} \cdot \alpha_k^{(t)}, \quad \beta_s^{(t)} = \alpha_h^{(t)} \cdot \beta_k^{(t)}.$$

It follows immediately from the shape of the eigenvectors in Lemma 3.1 that both $\alpha_h^{(t)} \cdot \alpha_k^{(t)} - u_h \cdot u_k$ and $\alpha_h^{(t)} \cdot \beta_k^{(t)} - u_h \cdot u_k$ are linear combinations of products of the form $a_r \cdot v$, where $r \in T$ and $v \in A \cup U$ and hence (iii) follows. $\square$

The next lemma is an important specialization of the above proposition.

Lemma 3.3 *With t, h, k as in Proposition 3.2 suppose that h and k commute. Then u_h and u_k are perpendicular and annihilate each other:*

$$(u_h, u_k) = u_h \cdot u_k = 0.$$

Proof. The subgroup $P = \langle t, h, k \rangle$ generated by t , h and k is a semi-direct product of the elementary abelian group Q of order 9 generated by h and k , and the order 2 group generated by t , where t inverts every element of Q . The group P possesses a unique Majorana representation satisfying the (3A)-condition. The vector space underlying this representation is spanned by four 3A-axis vectors u_h , u_k , u_{hk} , $u_{hk^{-1}}$ together with nine Majorana axis vectors $a_{h^i k^j t k^{-j} h^{-i}}$ for $0 \leq i, j \leq 2$. Any pair of distinct Majorana axes generate a 3A-subalgebra and so do a Majorana axis together with a 3A-axis. Therefore, by Table 1, in the notation of Proposition 3.2 we have

$p = \frac{1}{4}$, $q = \frac{13}{2^6}$ and calculating with the formulae therein we obtain the orthogonality. The annihilation property requires longer calculations with products of eigenvectors, but this is still doable by hand and fairly straightforward. $\square$

The four 3A-axes and nine Majorana axes in the representation of the group $P \cong 3^2 : 2$ defined in the proof of Lemma 3.3 are not linearly independent. The vector

$$45(u_h + u_k + u_{hk} + u_{hk^{-1}}) - 32 \sum_{1 \leq i, j \leq 2} a_{h^i k^j t k^{-j} h^{-i}}$$

is the zero vector, since its length is zero and $(\cdot, \cdot)$, being an inner product, is positive definite. This relation was shown to me by Dima Pasechnik and hence I call it *Pasechnik's relation*.³ The relation has the following important consequence vital for our proof.

Lemma 3.4 *Suppose that h and k are disjoint 3-cycles in G . Then the sum $u_h + u_k$ is a linear combination of some Majorana axes and 3A-axial vectors u_g for g generated by products of two disjoint 3-cycles.* $\square$

We conclude this section with yet another specialization of Proposition 3.2 (i). It is easy to check that under conjugation by $\text{Aut}(A_6)$ there is a unique orbit on pairs (h, k) of elements of order 3 generating A_6 with representative

$$h = (1, 2, 3)(4, 5, 6), \quad k = (1, 2, 4)$$

(these two elements are simultaneously inverted by $t = (1, 2)(5, 6)$.)

Lemma 3.5 *With t, h, k as in Proposition 3.2 suppose that h and k generate A_6 . Then $(u_h, u_k) = \frac{32}{405}$.*

Proof. By the paragraph just before the lemma we assume that $h = (1, 2, 3)(4, 5, 6)$, $k = (1, 2, 4)$, and $t = (1, 2)(5, 6)$. We apply Proposition 3.2 (i). The product of h and ktk^{-1} has order 5 which gives $p = \frac{1}{18}$ (cf. the table in the proof of Proposition 2.7 (ii)). The products of hth^{-1} with ktk^{-1} and $k^{-1}tk$ have orders 5 and 4, respectively. By Table 1 this gives $q = \frac{3}{2^7} + \frac{1}{2^6} = \frac{5}{2^7}$. Now the formulae in Proposition 3.2 (i) gives the claimed value for the inner product. $\square$

Finding an explicit formulae for the product $u_h \cdot u_k$ for a generating pair of 3-elements appears far too complicated to be calculated by hand.

The following lemma can also be proved by the methods developed in this section.

³The referee has pointed out that this relation is actually known in VOA theory since $u_h + u_k + u_{hk} + u_{hk^{-1}}$ and $\sum_{1 \leq i, j \leq 2} a_{h^i k^j t k^{-j} h^{-i}}$ are both scalar multiples of the Virasoro element.

Lemma 3.6 *The following assertions hold:*

- (i) *if $h = (1, 2, 3)(4, 5, 6)$ and $k = (1, 2, 7)$ then $(u_h, u_k) = \frac{64}{405}$;*
- (ii) *if $h = (1, 2, 3)(4, 5, 6)$ and $k = (1, 4, 7)$ then $(u_h, u_k) = \frac{32}{405}$. $\square$*

Note that Lemmas 3.3, 3.5 and 3.6 contains representatives of all the orbits of A_7 on pairs of order 3 subgroups with one being generated by a single 3-cycle and the other one by two such cycles.

4 Proof of Proposition 2.7 (iii)

We consider the two alternating groups in separate subsections.

4.1 A_6 -case

By Corollary 2.8 and Proposition 3.2 (iii) the assertion would follow if we can show that any pair of elements of order 3 in G are simultaneously inverted by an involution. This is indeed the case when $G \cong A_6$.

Lemma 4.1 *Any two elements h and k of order 3 in A_6 are simultaneously inverted by an involution.*

Proof. This is indeed a very elementary fact. A possible argument is as follows. First suppose that h and k are conjugate in A_6 . Then taking into account the outer automorphisms of A_6 we may assume that both h and k are 3-cycles (and certainly distinct). Then up to conjugation and inversion we just have the following three possibilities, where t is the required involution:

$$h = (1, 2, 3), \quad k = (1, 2, 4), \quad t = (1, 2)(5, 6);$$

$$h = (1, 2, 3), \quad k = (1, 4, 5), \quad t = (2, 3)(4, 5);$$

$$h = (1, 2, 3), \quad k = (4, 5, 6), \quad t = (1, 2)(4, 5).$$

If h and k are from different conjugacy classes, then they conjugate to the pair given just before Lemma 3.5, which is also simultaneously inverted as shown there, or to $(1, 2, 3)(4, 5, 6)$ and $(1, 2, 3)$ which are simultaneously inverted by $(1, 2)(4, 5)$. $\square$

4.2 A_7 -case

The analogue of Lemma 4.1 fails for A_7 , since this group contains the following elements which are not simultaneously inverted by an involution of A_7 :

$$h = (1, 2, 3)(4, 5, 6)(7), \quad k = (1, 2, 4)(3, 5, 7)(6).$$

(in fact there is not even an involution centralizing one and inverting the other, which would enable us to apply another version of the resurrection principle). On the other hand, this pair generates $L_3(2)$, rather than the whole of A_7 , and hence Lemma 2.3 is applicable. Let us get it all organized.

We start by applying Pasechnik's relation in the form of Lemma 3.4. For $i = 1$ or 2 put

$$U^{(i)} = \{u_h \mid h \in G^{(3)} \text{ and } h \text{ has } i \text{ cycles}\}.$$

Lemma 4.2 $\langle A \cup U^{(2)} \rangle = \langle A \cup U \rangle$.

Proof. Let

$$h_1 = (1, 2, 3), \quad h_2 = (4, 5, 6), \quad h_3 = (1, 2, 7), \quad h_4 = (3, 4, 6),$$

$$h_5 = (2, 5, 7), \quad h_6 = (1, 3, 4), \quad h_7 = (5, 6, 7), \quad h_8 = h_1.$$

If $1 \leq i \leq 6$ then $[h_i, h_{i+1}] = 1$ and by Lemma 3.4 $(u_{h_i} + u_{h_{i+1}}) \in \langle A \cup U^{(2)} \rangle$. Now the obvious the equality

$$\sum_{1 \leq i \leq 7} (-1)^i (u_{h_i} + u_{h_{i+1}}) = -2u_{h_1}$$

proves the assertion. □

The finishing touch of this subsection is the following lemma.

Lemma 4.3 *The action of S_7 on the set of subgroups of order 3 in A_7 generated by products of two 3-cycles is multiplicity-free of degree 10. A pair (h, k) of such order 3 elements (up to reordering and inverting) corresponds to a row in the following table (the entry in the fifth column indicates whether h and k are simultaneously inverted by an involution of A_7).*

#	$o(hk)$	$o(hk^{-1})$	$\langle h, k \rangle$	$\pm$	(u_h, u_k)
1	1	3	Z_3	+	$\frac{8}{5}$
2	3	3	$Z_3 \times Z_3$	+	0
3, 4	2	3	A_4	+	$\frac{136}{405}$
5	3	7	F_{21}	−	$\frac{4}{27}$
6	3	6	$A_4 \times Z_3$	+	$\frac{64}{405}$
7	5	5	A_5	+	$\frac{16}{405}$
8	4	4	$L_3(2)$	+	$\frac{32}{405}$
9	4	7	$L_3(2)$	−	$\frac{4}{81}$
10	5	6	A_7	+	$\frac{8}{81}$

The group A_7 contains two classes of A_4 -subgroups generated by a pairs of elements of order 3 having two 3-cycles: they preserve partitions $7=1+6$ and $7=3+4$, respectively. This explains the entry 3, 4 in the first column of the above table.

Proof. The classification of pairs (h, k) in question is fairly elementary and can be achieved by hand calculations, although computer calculations proved to be more efficient. The permutation character has been computed by Dima Pasechnik and Igor Faradjev has observed that the pair $\{o(hk), o(hk^{-1})\}$ of product orders determine the isomorphism type of the generated subgroup. Then Dima listed the representatives of the ten orbits on pairs and determined the isomorphism type of the subgroups they generate. This information is collected in the first five columns of the above table. Let us turn to the last column containing the inner product. In the terminology of [ISh10] the orbits numbered 5, 8 and 9 correspond to pairs of 3-, 2- and 4-related anti-flags in the relevant projective plane of order 2.

After Lemmas 2.3 and 3.3 are applied and the inner products calculated in [IPSS10], [ISe10] and [ISh10] are inserted, we are left with two cases to deal with: the orbits numbered 6 and 10. In each of these two cases there is an involution simultaneously inverting the elements of order 3 and hence Proposition 3.2 (i) is applicable.

The calculations are similar to those in the proof of Lemma 3.3. The details are as follows.

The pair $h = (1, 2, 3)(4, 5, 6)$, $k = (1, 2, 7)(4, 5, 6)$ represents orbit number 6 and $t = (1, 2)(4, 5)$ inverts both h and k . In terms of Proposition 3.2 (i) we have $p = \frac{1}{36}$, $q = \frac{9}{27}$, which gives $(u_h, u_k) = \frac{64}{405}$. For orbit number 10 we have $h = (1, 2, 3)(4, 5, 6)$, $k = (1, 2, 4)(5, 6, 7)$, $t = (1, 2)(5, 6)$, $p = \frac{1}{24}$, $q = \frac{11}{28}$, which gives $(u_h, u_k) = \frac{1}{81}$. $\square$

Proof of Proposition 2.7, completion. To prove assertion (iii) first recall that by Corollary 2.8 $\langle A \cup U \cup W \rangle$ is Majorana stable. Let $h, k \in G^{(3)}$. If $G \cong A_6$ then by Lemma 4.1 h and k are simultaneously inverted by an involution t of A_6 and Proposition 3.2 (iii) is applicable. If $G \cong A_7$ then by Lemma 4.2 we can assume that $u_h, u_k \in U^{(2)}$, so that h and k are generated by products of two 3-cycles. By Lemma 4.3 and the table therein we have one of the following two possibilities: either h and k are contained in a subgroup H with

$$H \in \{D_{12}, A_4, S_4, A_5, L_3(2), F_{21}\},$$

or h and k are simultaneously inverted by an involution of A_7 . In the former case we apply Lemmas 2.3 and 2.4, and in the latter case Proposition 3.2 (iii) can be applied exactly as in the A_6 -case discussed above. The situation when h and k commute is a particular case of the second possibility (in this case the product $u_h \cdot u_k$ is zero by Lemma 3.3). This completes the proof of assertion (iii).

Assertion (iv) now follows from the observation that the products in $V(H)$ with H as in the previous paragraph, and also the products obtained via Proposition 3.2 (iii), are uniquely determined by the factors. $\square$

It is worth mentioning that *a priori* we did not assume that the Majorana representations of A_6 and A_7 are stable under $\text{Aut}(A_6)$ and S_7 , but this can be seen *a posteriori*.

5 Dimensions and Monstralizers

There is a standard procedure for obtaining an upper bound for the dimension of a Majorana representation of a group G based on its embedding in the Monster (cf. Lemma 2.2 in [ISe10]).

Lemma 5.1 *Let M be the Monster group, let $\Pi = \Pi_0 \oplus \Pi_1$ be the vector space underlying the Monster algebra, where Π_0 is 1-dimensional trivial M -module, while Π_1 is minimal faithful of dimension 196 883. Let A_6 and A_7 be alternating subgroups in M generated by 2A-involutions and let $M_6 = C_M(A_6)$ and $M_7 = C_M(A_7)$. Then*

$$d_6 := \dim C_{\Pi_1}(M_6) = 77 \text{ and } d_7 := \dim C_{\Pi_1}(M_7) = 204.$$

Proof. The Monstralizers M_6 and M_7 are contained in the maximal subgroups of the Monster of shape

$$(A_6 \times A_6 \times A_6).(2 \times S_4) \text{ and } (A_7 \times A_5 \times A_5).D_8,$$

respectively. The dimension d_6 has been calculated by independent methods by Simon Norton and Dima Pasechnik, while d_7 was calculated by Sergey Shpectorov. $\square$

Since M_6 centralizes the set $A := \{a_t \mid t \in A_6\}$ of Majorana axes, it centralizes $\ll A \gg$ (where the closure is in the Monster algebra). Hence the dimension of the Majorana representation of A_6 based on the embedding in the Monster is at most $78 = d_6 + \dim \Pi_0$ and similarly for A_7 . On the other hand, the dimensions are equal to the ranks of the Gram matrices of the corresponding spanning sets and thus can be calculated precisely.

Proposition 5.2 *Let G be isomorphic to the alternating group of degree 6 or 7 and let $\mathcal{R} = (G, T, V, (\ , \), \cdot, \varphi, \psi)$ be a Majorana representation of G satisfying the hypothesis of Theorem 1. Let $A = \{a_t \mid t \in T\}$ be the set of Majorana axes and let $U = \{u_h \mid h \in G^{(3)}\}$ be the set of 3A-axial vectors. Then*

- (i) $\mathcal{R}$ is uniquely determined by G , based on embedding of G in the Monster, and $\text{Aut}(G)$ -stable;
- (ii) $\langle A \cup U \rangle$ has co-dimension 0 or 1 in V ;
- (iii) if $G \cong A_6$ then $\dim(\langle A \cup U \rangle) = 75$;
- (iv) if $G \cong A_7$ then $\dim(\langle A \cup U \rangle) = 196$.

Proof. Assertion (i) follows from Proposition 2.7 (iii), (iv), Lemma 2.6 and the fact that the Monster contains 2A-generated subgroups A_6 and A_7 . Assertion (ii) is by Lemma 2.5. Assertion (iii) has been established by Dima Pasechnik. The details are as follows: there are 10 Pasechnik relations, one for every Sylow 3-subgroup in A_6 . These relations were shown to be linearly independent and it turned out that there are no further relations other than linear combinations of these 10. Hence

$$\dim(\langle A(A_6) \cup U(A_6) \rangle) = |A(A_6)| + |U(A_6)| - 10 = 45 + 40 - 10 = 75.$$

In case (iii) the calculations were performed by Igor Faradjev. Each of the sets $A(A_7)$ and $U(A_7)$ turned out to be linearly independent and hence it spans in $V(A_7)$ the permutation module of A_7 (or rather of S_7) on involutions and order 3 subgroups generated by pairs of 3-cycles. By the upper bound in Lemma 5.1 the whole of

$A(A_7) \cup U^{(2)}(A_7)$ cannot be linearly independent since it contains $105+140=245$ vectors. Specifically, it was shown that inside $V(A_7)$ the permutation module on $A(A_7)$ and the permutation module on $U^{(2)}(A_7)$ intersect in a 49-dimensional subspace, which is the direct sum of a 14- and a 35-dimensional A_7 -modules. Thus

$$\dim(< A(A_7) \cup U(A_7) >) = |A(A_7)| + |U^{(2)}(A_7)| - 49 = 196$$

and there are 49 linearly independent Faradjev relations in $V(A_7)$. $\square$

By Proposition 5.2 and Lemma 2.5 assertions (iii) and (iv) of Theorem 1 follow.

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