Newhouse regions for reversible systems with infinitely many stable, unstable, and elliptic periodic orbits

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Abstract

For reversible two-dimensional diffeomorphisms we establish a new type of Newhouse regions (regions of structural instability density). We prove that in these regions there exists a dense set of diffeomorphisms having simultaneously infinitely many stable, infinitely many unstable and infinitely many elliptic type periodic orbits.

1 Introduction and statement of the main results.

One of the most important results of bifurcation theory of dynamical systems is the discovery of regions of structural instability in the space of smooth systems. Examples of these regions are the regions in which there exists a dense set of systems with saddle periodic orbits having tangencies of their stable and

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unstable manifolds along homoclinic orbits. This fact was established by Newhouse [27, 26] in the case of two-dimensional diffeomorphisms. Later, this result was extended to higher dimensions in [17, 28, 30]. Naturally, such regions, both in the space of dynamical systems and in the parameter space, have been called Newhouse regions.

Dynamical properties of systems in Newhouse regions are very complicated. In order to describe the bifurcation phenomena of such systems, infinitely many parameters are required. This is connected with the fact that in Newhouse regions diffeomorphisms with periodic (and homoclinic) orbits of any order of degeneracy are dense [19, 18, 13, 16]. Another important property of systems in Newhouse regions is the presence of infinitely many stable (or unstable) periodic orbits [5, 25, 18]. Note, that such type Newhouse regions appear in generic one-parameter families which are transverse to a codimension one bifurcation surface, consisting of diffeomorphisms with homoclinic tangencies of manifolds of dissipative saddle fixed points.

An essentially other type of Newhouse regions arises near diffeomorphisms with a non-transversal heteroclinic cycle consisting of several saddle fixed (periodic) points and heteroclinic orbits. In the (codimension one) case when the heteroclinic cycle contains only one non-transversal heteroclinic orbit with a quadratic tangency of corresponding stable and unstable manifolds and the saddle values of one of fixed points is less than one, and the saddle value of another is greater than one, a new phenomenon was established [20]: there exist Newhouse regions in which diffeomorphisms that have simultaneously infinitely many saddle, stable and completely unstable periodic orbits are dense. This assertion also holds for generic one parameter families. Moreover, in the context of two parameter analysis allows to prove the density of diffeomorphisms with infinitely many stable and unstable invariant closed curves [12]. In these situations, it is naturally to speak of Newhouse regions (intervals) *with mixed dynamics*. Note that in these regions diffeomorphisms with homoclinic tangencies are also dense and, consequently, diffeomorphisms with above mentioned structurally unstable periodic and homoclinic orbits are dense.

The aim of the present paper is to describe the properties of Newhouse regions which can exist in two-dimensional reversible diffeomorphisms. Recall that a diffeomorphism $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is called reversible if there exists a diffeomorphism $R : \mathbb{R}^2 \mapsto \mathbb{R}^2$, which conjugates f to its inverse f^{-1} , ie

$$R \circ f \circ R^{-1} = f^{-1},$$

see for instance [22].

In this paper, we will confine ourselves to the case that $R^2 = I$, f is orientation preserving ($\det df > 0$), and R is orientation reversing ($\det dR < 0$). Let $\text{Fix}(R)$ denote the set of fixed points of R . It follows from our assumptions that $\text{Fix}(R)$ and $\text{Fix}(f \circ R)$ have dimension 1. For such reversible two-dimensional diffeomorphisms it is well known that symmetric (R -invariant) periodic orbits are generically either of saddle type (one attracting and one repelling multiplier) or elliptic (two multipliers on the complex unit circle). Asymmetric periodic orbits arise pairwise: they either form R -invariant pairs of stable and completely unstable periodic orbits, or pairs of saddle periodic orbits (where typically the Jacobian is greater than 1 for one of them and less than 1 for the other one).

We consider generic one parameter families of two-dimensional reversible diffeomorphisms containing a diffeomorphism with a non-transversal heteroclinic cycle consisting of two saddle fixed points and two heteroclinic orbits, one of which is non-transversal. We show that in any parameter neighborhood of the parameter value corresponding to the non-transversal heteroclinic cycle, there are Newhouse parameter intervals where diffeomorphisms having simultaneously infinitely many stable, completely unstable, saddle and elliptic type periodic points, are dense.

We now make our assumptions more precise. Let f_μ be a one (μ) parameter family of planar C^r -smooth ($r \geq 5$) R -reversible diffeomorphisms, where R is orientation reversing and f orientation preserving. We suppose that

- [H1] f_μ has a saddle fixed point $O_1 \notin \text{Fix}(R)$, with multipliers λ, γ , $|\lambda| < 1 < |\gamma|$. It follows from reversibility that f has another saddle fixed point $O_2 = R(O_1)$ with multipliers $\gamma^{-1}, \lambda^{-1}$, $|\gamma^{-1}| < 1 < |\lambda^{-1}|$.

[H2] We suppose that the Jacobians $\sigma_i = \det(df_0(O_i))$, with $i = 1, 2$, are not equal to 1. Note that $\sigma_1 = \lambda\gamma$; $\sigma_2 = (\lambda\gamma)^{-1}$.

[H3] The diffeomorphism f_0 has two heteroclinic orbits, Γ_{12} and Γ_{21} , such that $\Gamma_{12} \subset W^u(O_1) \cap W^s(O_2)$ and $\Gamma_{21} \subset W^u(O_2) \cap W^s(O_1)$. We suppose that

- 1) $W^u(O_1)$ and $W^s(O_2)$ intersect transversally at the points of orbit Γ_{12}
- 2) $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency at the points of orbit Γ_{21} ,
- 3) $R(\Gamma_{12}) = \Gamma_{12}$, $R(\Gamma_{21}) = \Gamma_{21}$.

The set $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$ is a heteroclinic cycle. We call such a cycle a *non-transversal symmetric heteroclinic cycle*, as it is non-transversal by 2) and R -invariant by 3). There turn out to be four different types of such cycles. For the case of positive multipliers they are sketched in Fig. 1.

The diffeomorphisms which are close to f_0 and satisfy the conditions [H1-H3] form a locally connected surface H of codimension 1 in the space of reversible two-dimensional diffeomorphisms. In this paper we study one-parameter families f_μ , where μ is a parameter governing the splitting of the manifolds $W^u(O_2(\mu_1))$ and $W^s(O_1(\mu_1))$. We thus assume that

[H4] f_μ is transversal to H .

Our main result is summarized in the following theorem:

Theorem 1 *Let f_μ be a one parameter family of R -reversible planar diffeomorphisms as discussed above, satisfying [H1-H4]. Then, for any $\mu_0 > 0$, the interval $I = (-\mu_0, \mu_0)$ contains a countable set of intervals $\delta_j \subset I$ of values of μ such that $\delta_j \rightarrow 0$ as $j \rightarrow \infty$ and*

- 1) *in δ_j there is dense set of values of μ , $\mu = \mu^*$, at which the diffeomorphism f_μ has a non-transversal homoclinic orbit $\Gamma_1(\mu)$ to O_1 (and consequently also a non-transversal homoclinic orbit $\Gamma_2(\mu) = R\Gamma_1(\mu)$ to $O_2 = R(O_1)$). Moreover, the parameter $\mu - \mu^*$ splits this tangency monotonically.*
- 2) *in δ_j there is a dense set of values of μ , $\mu = \mu^*$, at which the diffeomorphism f_{μ^*} has a symmetric type non-transversal heteroclinic cycle consisting of fixed points $O_1(\mu^*)$, $O_2(\mu^*)$, a transversal heteroclinic orbit $\Gamma_{12}(\mu^*)$ and a non-transversal heteroclinic orbit $\Gamma_{21}^j(\mu^*)$ in whose points the manifolds $W^u(O_2(\mu^*))$ and $W^s(O_1(\mu^*))$ have a quadratic tangency; the parameter $\mu - \mu^*$ splits this tangency monotonically.*
- 3) *in δ_j there is a dense set of values of μ , $\mu = \mu^*$, at which diffeomorphism f_{μ^*} has a non-symmetric type non-transversal heteroclinic cycle consisting of orbits $O_1(\mu^*)$, $O_2(\mu^*)$, $\Gamma_{12}(\mu^*)$ and $\Gamma_{21}^j(\mu^*)$; the parameter $\mu - \mu^*$ splits this tangency monotonically.*
- 4) *in δ_j there is a dense set of values of μ , at which diffeomorphism f_μ has infinitely many saddle, infinitely many stable, infinitely many completely unstable periodic orbits and infinitely many elliptic type periodic points, whose closure contains the fixed points O_1 and O_2 .*

In analogy to Newhouse's result on the persistence of homoclinic tangencies [26, 17], for the one parameter families under consideration we find a dense set of diffeomorphisms with homoclinic tangencies in parameter intervals δ_j . Accordingly, we call δ_j *Newhouse intervals*. However, we prove that in our context of reversible systems, the dynamics of the diffeomorphisms is mixed, with elliptic, stable, and unstable behaviour.

Note that in area-preserving case dynamical properties of diffeomorphisms in Newhouse regions have been studied by Duarte [4]. In particular, in [3], the existence of infinitely many elliptic orbits were found for area-preserving Henon maps. Gonchenko, Shilnikov and Turaev [14, 15] discuss generic 2-elliptic

Figure 1: Various types of diffeomorphism with non-transversal heteroclinic cycle

periodic orbits at four-dimensional symplectic maps close to a map with a non-transversal homoclinic orbit to a saddle-focus fixed point. The phenomenon of "mixed dynamics" for area-preserving two-dimensional diffeomorphisms was also discussed in [10, 11].

To our knowledge, our paper is the first to discuss Newhouse regions for reversible diffeomorphisms, revealing the novel phenomenon of the simultaneous occurrence of infinitely many stable, completely unstable and elliptic periodic orbits. Importantly, although it is well known that elliptic, saddle, attracting and repelling periodic points can coexist in reversible diffeomorphisms, it is far from obvious that infinities of those should coexist persistently in the neighbourhood of symmetric heteroclinic tangencies. Moreover, by the simultaneous creation of symmetric (saddle, elliptic) and nonsymmetric (attractors, repellers) periodic orbits, the heteroclinic bifurcation thus induces some degree of symmetry breaking, by having close to the non-transversal cycle both symmetric and nonsymmetric orbits, as well as nonsymmetric tangencies.

In the literature, homoclinic and heteroclinic orbits in reversible systems have most frequently been studied in the context of ODEs, where the connecting orbits arise between equilibria, see for instance [2, 21, 23]. Our study relates to the study of dynamics near heteroclinic orbits in reversible systems between a pair of asymmetric periodic solutions of saddle type. In fact, an important class of reversible ODEs are those derived from a $\mathbb{Z}_2$ -symmetric PDE [2, 21, 23], by restriction to travelling and/or standing waves. In this context, our results can be interpreted to shed light on the complicated structure of the solution set of the PDE in the neighbourhood of heteroclinic cycles between periodic waves.

Two-dimensional reversible diffeomorphisms appear naturally as Poincaré maps for symmetric periodic orbits in three-dimensional reversible vector fields. Such a situation arises for instance in the travelling/standing wave reduction of the Kuramoto-Shivashinsky PDE [24, 23]. Alternatively, such reversible maps can appear naturally when periodically forced reversible flows are considered where the forcing has a special form (eg odd or even function of time).

We now proceed with discussing yet another example from the recent physics literature where reversible two-dimensional maps having non-transversal heteroclinic cycles appear as Poincaré maps. Topaj and Pikovsky [29] describe the dynamics of four coupled rotators, by the following set of differential equations

$$\begin{aligned}\dot{\psi}_1 &= 1 - 2\varepsilon \sin \psi_1 + \varepsilon \sin \psi_2 \\ \dot{\psi}_2 &= 1 - 2\varepsilon \sin \psi_2 + \varepsilon \sin \psi_1 + \varepsilon \sin \psi_3 \\ \dot{\psi}_3 &= 1 - 2\varepsilon \sin \psi_3 + \varepsilon \sin \psi_2\end{aligned}\tag{1}$$

where the ψ_i denote phase differences and ε is the coupling coefficient. This system is reversible with respect to the involution $(\psi_1, \psi_2, \psi_3) \rightarrow (\pi - \psi_3, \pi - \psi_2, \pi - \psi_1)$. When ε is small, the three-dimensional flow (1) can be reduced to the following non-autonomous periodically forced two-dimensional flow

$$\begin{aligned}\dot{\xi} &= \frac{2\varepsilon \sin \xi \sin \eta}{2 + \varepsilon \cos(t - \eta)} \\ \dot{\eta} &= \frac{1 - \varepsilon \cos(t - \eta) - 2\varepsilon \cos \xi \cos \eta}{2 + \varepsilon \cos(t - \eta)}\end{aligned}$$

which is reversible with respect to the involution $(\xi, \eta) \rightarrow (\xi, -\eta)$. Numerical experiments from [29, 6] indicates that the Poincaré map can display a heteroclinic cycle with visible traces of the Newhouse phenomenon, as is studied in detail in the present paper. Illustrations in [29] may be interpreted as the existence of "intersection" of attractors and repellers, i.e. the closure of sets of stable and completely unstable periodic orbits containig hyperbolic subsets. For more on this, see [29, 6]).

This paper is organized as follows. In Section 2 we discuss the geometric construction underlying our proof of Theorem 1. We discuss also the classification of diffeomorphisms with a non-transversal heteroclinic cycle and point out the principal difference with the non-reversible case. To study the main bifurcations of such diffeomorphisms we consider general one-parameter unfoldings. In Section 3 we prove the statements 1 – 3 of Theorem 1. In Section 4 we prove the first three statements of Theorem 1.

Figure 2: The neighbourhood of a non-transversal symmetric heteroclinic cycle

In order to prove density of diffeomorphisms with mixed types of periodic orbits we consider bifurcations of single-round periodic orbits. In particular, in subsection 4.1 (the Rescaling Lemma) we reduce the study of the corresponding Poincaré map to the study of the reversible Henon map. It allows us to find diffeomorphisms with elliptic periodic points in the subsection 4.2. Subsection 4.3 is devoted to prove the existence of diffeomorphisms with stable and unstable periodic orbits. We reduce this problem to the study of bifurcations of a parabola map. Finally, in subsection 4.4 we summarize our results and prove the fourth and final statement of Theorem 1.

2 Geometric construction

We consider C^r -smooth ($r \geq 5$), two-dimensional reversible diffeomorphism f_0 having a symmetric non-transversal heteroclinic cycle C as detailed in Section 1. Consider a sufficiently small fixed neighbourhood U of the cycle C . U is a union of two discs U_1 and U_2 containing the points O_1 and O_2 , and a number of discs containing points of orbits Γ_{12} and Γ_{21} which lie outside U_1 and U_2 , see Fig. 2 for a sketch.

2.1 Local and global maps

Denote by $T_{0l}(\mu)$, $l = 1, 2$, the restriction of f_μ onto U_l , i.e. $T_{0l}(\mu) \equiv f_\mu|_{U_l}$. We will call maps $T_{0l}(\mu)$ and $T_{02}(\mu)$ the *local maps*. Consider the neighbourhood U_1 . It is known [8, 9, 7] that there exist C^{r-1} -

coordinates on U_1 such that $T_{0l}(\mu)$ can be written in the form

$$\begin{aligned}\bar{x}_1 &= \lambda(\mu)x_1 + g_1(x_1, y_1, \mu)x_1^2 y_1, \\ \bar{y}_1 &= \gamma(\mu)y_1 + g_2(x_1, y_1, \mu)x_1 y_1^2,\end{aligned}\tag{2}$$

where the right sides of (2) are C^{r-1} (in coordinates and parameters). In these coordinates, for sufficiently small μ , the point $O_1(\mu)$ has coordinates $x_1 = 0, y_1 = 0$ and the equations of $W_{loc}^s(O_1(\mu))$ and $W_{loc}^u(O_1(\mu))$ are $y_1 = 0$ and $x_1 = 0$, respectively. Of course, the coordinates (x_1, y_1) are only defined in the small region U_1 of O_1 . Let (u, v) denote the natural global coordinates in $\mathbb{R}^2$. Then there is a coordinate transformation $q : U_1 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} u \\ v \end{pmatrix} = q \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

The representation (2) of T_{01} in the local coordinates (x_1, y_1) is given by

$$T_{01} = q^{-1} \circ f \circ q$$

Now we choose coordinates (x_2, y_2) in U_2 such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \tilde{q} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

where $\tilde{q} : U_2 \rightarrow \mathbb{R}^2$, $\tilde{q} = R \circ q \circ R_{12}^{-1}$, and $R_{12} : U_1 \rightarrow U_2$ is some smooth diffeomorphism. By reversibility of f we have $R^{-1} \circ f^{-1} \circ R = f$, so that

$$T_{01} = q^{-1} \circ f \circ q = q^{-1} \circ R^{-1} \circ f^{-1} \circ R \circ q = R_{12}^{-1} \circ \tilde{q}^{-1} f^{-1} \circ \tilde{q} \circ R_{12}.$$

Analogously, in local coordinates we have

$$T_{02} = \tilde{q}^{-1} \circ f \circ \tilde{q},$$

and hence

$$T_{01} = R_{12}^{-1} \circ T_{02}^{-1} \circ R_{12}.\tag{3}$$

It is convenient to choose R_{12} as $R_{12}(x_1, y_1) = (y_2, x_2)$. By virtue of (2) and (3), the local map $T_{02}^{-1}(\mu)$ will thus have the form

$$\begin{aligned}y_2 &= \lambda(\mu)\bar{y}_2 + f(\bar{y}_2, \bar{x}_2, \mu)\bar{y}_2^2 \bar{x}_2, \\ x_2 &= \gamma(\mu)\bar{x}_2 + g(\bar{y}_2, \bar{x}_2, \mu)\bar{y}_2 \bar{x}_2^2,\end{aligned}\tag{4}$$

Consider again the diffeomorphism f_0 . The point O_1 is an α -limit point for Γ_{12} and an ω -limit point for Γ_{21} ; the point O_2 is an α -limit point for Γ_{21} and an ω -limit point for Γ_{12} . Hence, countable sets of heteroclinic points (points of orbits Γ_{12} and Γ_{21}) lie on $W_{loc}^s(O_l)$ and $W_{loc}^u(O_l)$. Since Γ_{12} and Γ_{21} are symmetric, i.e. $R(\Gamma_{12}) = \Gamma_{12}$ and $R(\Gamma_{21}) = \Gamma_{21}$ we can choose two pairs of the symmetric heteroclinic points: $M_2^+(x_2^+, 0) \in U_2$ and $M_1^-(0, y_1^-) \in U_1$ of the orbit Γ_{12} and $M_1^+(x_1^+, 0) \in U_1$ and $M_2^-(0, y_2^-) \in U_2$ of the orbit Γ_{21} , where $x_1^+ = y_2^-$ and $x_2^+ = y_1^-$. Without loss of generality, we may choose $x_1^+ > 0$. Let n_1 and n_2 be positive integers such that $f_0^{n_1}(M_1^-) = M_2^+$ and $f_0^{n_2}(M_2^-) = M_1^+$. Consider sufficiently small neighbourhoods $\Pi_l^+ \subset U_l$ and $\Pi_l^- \subset U_l$ of points M_l^+ and M_l^- , respectively. Then, the *global maps* $T_{12}(\mu) \equiv f_\mu^{n_1} : \Pi_1^- \rightarrow U_2$ and $T_{21}(\mu) \equiv f_\mu^{n_2} : \Pi_2^- \rightarrow U_1$ are defined for all μ small.

Denote coordinates on Π_l^+ and Π_l^- as (x_{0l}, y_{0l}) and (x_{1l}, y_{1l}) , respectively. Then, the global maps can be written as follows. Map $T_{12}(\mu)$ has the following form (the Taylor expansion near the point $x_{11} = 0, y_{11} = y_1^-(\mu)$)

$$\begin{aligned}\bar{x}_{02} - x_2^+(\mu) &= a_{12}(\mu)x_{11} + b_{12}(\mu)(y_{11} - y_1^-(\mu)) + \dots \\ \bar{y}_{02} &= c_{12}(\mu)x_{11} + d_{12}(\mu)(y_{11} - y_1^-(\mu)) + \dots\end{aligned}\tag{5}$$

where coefficients $a_{12}, \dots, d_{12}$ depend on μ , in general; $x_2^+(0) = x_2^+$, $y_1^-(0) = y_1^-$. Note that points $(x_2^+(\mu), 0)$ and $(0, y_1^-(\mu))$ are the intersection points of the orbit $\Gamma_{12}(\mu)$ with Π_2^+ and Π_1^- , where by transversality of the intersection of stable and unstable manifolds $d_{12}(0) \neq 0$ and $x_2^+(\mu) = y_1^-(\mu)$ for all μ . Moreover, the Jacobian $J_{12} \equiv a_{12}d_{12} - b_{12}c_{12}$ of the map T_{12} is calculated in the point M_1^- at $\mu = 0$ is nonzero since T_{12} is a diffeomorphism.

It should be noted that reversibility imposes some conditions on the expressions in the right hand side of (5). Namely, consider the map $T_{12} : \Pi_1^- \rightarrow \Pi_2^+$, where $\Pi_1^- \in U_1$, $\Pi_2^+ \in U_2$. In local coordinates, we have

$$T_{12} = \tilde{q}^{-1} \circ f^{n_2} \circ q = R_{12} \circ q^{-1} \circ R^{-1} \circ f^{n_2} \circ q.$$

It is easy to check that

$$R_{12}^{-1} \circ T_{12} \circ R_{12}^{-1} = T_{12}^{-1}. \quad (6)$$

In order to simplify we rewrite formulas of (5) in the form

$$\begin{aligned} \bar{x}_{02} - x_2^+(\mu) &= F(x_{11}, y_{11} - y_1^-(\mu), \mu), \\ \bar{y}_{02} &= G(x_{11}, y_{11} - y_1^-(\mu), \mu), \end{aligned} \quad (7)$$

It follows from (6) and uniqueness $x_2^+(\mu) = y_1^-(\mu)$ that T_{12}^{-1} has the following form

$$\begin{aligned} x_{11} &= G(\bar{y}_{02}, \bar{x}_{02} - x_2^+(\mu), \mu), \\ y_{11} - y_1^-(\mu) &= F(\bar{y}_{02}, \bar{x}_{02} - x_2^+(\mu), \mu). \end{aligned} \quad (8)$$

Let us substitute the expressions for $\bar{x}_{02} - x_2^+(\mu)$ and $\bar{y}_{02}$ given by (7) into right side of (8). We obtain uniqueness

$$\begin{aligned} x_{11} &= G(G(x_{11}, y_{11} - y_1^-(\mu), \mu), F(x_{11}, y_{11} - y_1^-(\mu), \mu), \mu), \\ y_{11} - y_1^-(\mu) &= F(G(x_{11}, y_{11} - y_1^-(\mu), \mu), F(x_{11}, y_{11} - y_1^-(\mu), \mu), \mu). \end{aligned} \quad (9)$$

or

$$\begin{aligned} x_{11} &= (c_{12}^2 + a_{12}d_{12})x_{11} + d_{12}(b_{12} + c_{12})(y_{11} - y_1^-) + \dots, \\ y_{11} - y_1^- &= a_{12}(b_{12} + c_{12})x_{11} + (b_{12}^2 + a_{12}d_{12})(y_{11} - y_1^-) + \dots \end{aligned} \quad (10)$$

From this and $d_{12} \neq 0$, it follows that

$$b_{12}(\mu) = -c_{12}(\mu), \quad a_{12}(\mu)d_{12}(\mu) + b_{12}^2(\mu) = 1. \quad (11)$$

Of course, there exist special relations on coefficients of high order terms also.

The following lemma describes how the reversibility influences the form of the global map.

Lemma 1 *The map T_{12} can be written implicitly (in cross coordinates) as*

$$\begin{aligned} \bar{x}_{02} - x_2^+(\mu) &= \Phi(x_{11}, \bar{y}_{02}, \mu), \\ y_{11} - y_1^- &= \Phi(\bar{y}_{02}, x_{11}, \mu). \end{aligned} \quad (12)$$

where

$$\Phi(u, v, \mu) = A(\mu)u + B(\mu)v + O(u^2 + |uv| + v^2), \quad B(\mu) \neq 0 \quad (13)$$

Proof Note that the equation $w = G(u, v, \mu)$ define variable v uniquely as functions of u and v , i.e. $v = \Phi(u, w)$, because $\frac{\partial G}{\partial v}(0, 0, \mu) \neq 0$. It follows from the second equation of the system (7) and the first equation of (8) that $\bar{x}_{02} - x_2^+(\mu) = \Phi(x_{11}, \bar{y}_{02}, \mu)$ and $y_{11} - y_1^- = \Phi(\bar{y}_{02}, x_{11}, \mu)$. It is easy to check that $\Phi(u, v, \mu) = A(\mu)u + B(\mu)v + O(u^2 + |uv| + v^2)$, where $B(\mu) \neq 0$. $\blacksquare$

The map $T_{21}(\mu) \equiv f_\mu^{n_2} : \Pi_2^- \rightarrow U_1$ can be written in the following form (the Taylor expansion near the point $x_{12} = 0, y_{12} = y_2^-(\mu)$)

$$\begin{aligned} \bar{x}_{01} - x_1^+(\mu) &= E(\mu) + a_{21}(\mu)x_{12} + b_{21}(\mu)(y_{12} - y_2^-(\mu)) \dots \\ \bar{y}_{01} &= \mu + c_{21}(\mu)x_{12} + d_{21}(\mu)(y_{12} - y_2^-(\mu))^2 + \dots \end{aligned} \quad (14)$$

We now note that (R -equivariant) coordinate transformations that preserve the R -reversibility of the map, automatically preserve the relation $x_1^+(\mu) = y_2^-(\mu)$. We now may perform such a coordinate transformation so that $y_2^-(\mu)$ is such that the second equation of (14) does not contain a term linear in y_{12} , even when $\mu \neq 0$. However, unlike the situation in the nonreversible case [13, 20, 12], we will be left with a term $E(\mu)$ in the first equation of (14), which is generally nonzero if $\mu \neq 0$. We furthermore note that $d_{21}(0) \neq 0$ since $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency at point M_1^+ and $J_{21} \equiv -b_{21}(0)c_{21}(0) \neq 0$ since T_{21} is a diffeomorphism.

Note that μ enters in the second equation of (14) as a constant term that serves as the splitting parameter of the manifolds $W^u(O_2(\mu))$ and $W^s(O_1(\mu))$ at the point M_1^+ .

It follows from reversibility that

$$R_{12} \circ T_{21} \circ R_{12} = T_{21}^{-1}. \quad (15)$$

Hence, T_{21}^{-1} can be written as

$$\begin{aligned} y_{12} - y_2^-(\mu) &= E(\mu) + a_{21}(\mu)\bar{y}_{01} + b_{21}(\mu)(\bar{x}_{01} - x_1^+) \dots \\ x_{12} &= \mu + c_{21}(\mu)\bar{y}_{01} + d_{21}(\mu)(\bar{x}_{01} - x_1^+)^2 + \dots \end{aligned} \quad (16)$$

Like we did before for global map T_{12} (see formulaes (9) and (10)), we obtain the relations on coefficients of T_{21} . It is furthermore easy to check that

$$\begin{aligned} E(\mu) &= -a\mu/2 + O(\mu^2); & a_{21}(\mu) &= a + a_\mu\mu; \\ b_{21}(\mu) &= 1 + b_\mu\mu; & c_{21}(\mu) &= -1 + c_\mu\mu; \end{aligned} \quad (17)$$

For sufficiently large k and small μ we define the map $T_{01}^k(\mu) : \Pi_1^+ \rightarrow \Pi_2^-$. It can be written in the form [8, 9]

$$\begin{aligned} x_{11} &= \lambda(\mu)^k x_{01}(1 + \hat{\gamma}^{-k} p_k(x_{01}, y_{11}, \mu)) \\ y_{01} &= \gamma(\mu)^{-k} y_{11}(1 + \hat{\gamma}^{-k} q_k(x_{01}, y_{11}, \mu)) \end{aligned} \quad (18)$$

where $\hat{\gamma}^{-1} = \max\{\gamma^{-1}, \lambda\}$, and the functions p_k and q_k are uniformly bounded in k along with all derivatives up to order $(r-2)$. Thus, in these coordinates, for large k the map T_{01}^k will be asymptotically close to a linear map.

We recall from (3) that

$$T_{01}^k = R_{12}^{-1} \circ T_{02}^{-k} \circ R_{12}. \quad (19)$$

Hence, the map T_{02}^{-k} can be written in the form

$$\begin{aligned} y_{02} &= \lambda(\mu)^k y_{12}(1 + \hat{\gamma}^{-k} p_k(y_{12}, x_{02}, \mu)) \\ x_{12} &= \gamma(\mu)^{-k} x_{02}(1 + \hat{\gamma}^{-k} q_k(y_{12}, x_{02}, \mu)) \end{aligned} \quad (20)$$

2.2 Geometric Tools

Strips and Horseshoes.

We recall that U is a small neighborhood of $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$ and U consist of two small discs U_1 and U_2 containing points O_1 and O_2 , and a finite number of small discs surrounding the points of orbits Γ_{12} and Γ_{21} which are located outside U_1 and U_2 (Fig. 2). The subject of our study is the set $N(\mu)$ of orbits of the map $f(\mu)$ that lie entirely in U .

Consider the neighbourhood U_l , $l = 1, 2$. Note that orbits of N must intersect the neighbourhoods Π_l^+ and Π_l^- (otherwise, these orbits would be far from Γ). However, not all orbits that start in Π_l^+ arrive in Π_l^- . The set of the points whose orbits get into Π_l^- fills a countable number of strips $\sigma_k^{0l} = \Pi_l^+ \cap T_{0l}^{-k} \Pi_l^-$ which accumulate on $W_{\text{loc}}^s(O_l)$. The way of constructing these strips is obvious from Fig. 3a. In turn, the images of the strips σ_k^{0l} under the maps T_{0l}^k give on Π_l^- a sequence of strips σ_k^{1l} which accumulate on $W_{\text{loc}}^u(O_l)$ (Fig. 3b). This follows from the local representations of T_{0l}^k (18).

Figure 3: Method of a construction of the strips σ_k^{01} and σ_k^{11}

Neighborhoods Π_l^+ and Π_l^- may be taken so that to contain all the strips σ_k^{0l} and σ_k^{1l} with numbers $k \geq \bar{k}$, for some $\bar{k}$, and not to intersect with σ_k^{0l} and σ_k^{1l} for all $k < \bar{k}$. Obviously, if $\text{diam}\Pi_l^+ \cdot \text{diam}\Pi_l^- \rightarrow 0$, then $\bar{k} \rightarrow \infty$.

The images $T_{12}\sigma_k^{11}$ of the strips σ_k^{11} are strips accumulated at $T_{12}W_{\text{loc}}^u(O_1)$ as $k \rightarrow \infty$ (Fig. 4). It is clear that orbits of N must intersect Π_2^+ in points lying in intersections of strips $T_{12}\sigma_i^{11}$ and strips σ_j^{02} for $i, j \geq \bar{k}$.

Evaluating the action of the global map T_{21} , we note that the images $T_{21}\sigma_k^{12}$ of the strips σ_k^{12} have the shape of horse-shoes accumulating to $T_{21}W_{\text{loc}}^u(O_2)$ as $k \rightarrow \infty$ (Fig. 4). Of course, orbits of N must intersect Π_1^+ in points lying in intersections of horse-shoes $T_{21}\sigma_i^{12}$ and strips σ_j^{01} for $i, j \geq \bar{k}$. Hence, the structure of N depends essentially on the geometrical properties of such intersections.

The Types of Intersections of the Strips and Horseshoes.

The horse-shoe $T_{21}\sigma_i^{12}$ can have different types of intersections with the strips σ_j^{01} . We call the intersection *regular* if the set $T_{21}\sigma_i^{12} \cap \sigma_j^{01}$ is non-empty and consists of two connected components σ_{ji}^{01} and σ_{ji}^{02} . The different types of intersections are illustrated in Fig. 5: the horse-shoe has a regular intersection with the strip σ_j^{01} , an irregular intersection with the strip σ_k^{01} and empty intersection with the strip σ_l^{01} .

In [20], Gonchenko and Shilnikov established conditions for regular and irregular intersections of the strips and horseshoes. We may adapt these conditions to our case, taking into account the reversibility (eg $c_{21} = -1, \dots$).

Recall that $d_{21} \neq 0$ is the coefficient in (14) establishing the quadratic tangency. The sign of d_{21} signals whether the tangency is from “above” or “below”, with respect to the stable and unstable manifolds (see also Fig. 1).

If the inequality

$$d_{21}(\gamma^{-i}x_2^+ - \mu + \gamma^{-j}x_2^+) > S_{ij}(\bar{k}), \quad (21)$$

is satisfied, where $S_{ij} = S_1(|\gamma|^{-i} + |\gamma|^{-j})(|\lambda|^{\bar{k}} + |\gamma|^{-\bar{k}})$ and S_1 is some positive constant independent from i, j and $\bar{k}$, then the intersection of $T_{21}(\mu)(\sigma_j^{12})$ and the strip σ_i^{01} is regular.

The inequality

$$d_{21}(\gamma^{-i}x_2^+ - \mu + \gamma^{-j}x_2^+) < -S_{ij}(\bar{k}) \quad (22)$$

is a sufficient condition for an intersection of $T_{21}(\mu)(\sigma_j^{12})$ and σ_i^{01} to be empty.

It is clear from (21) and (22) that the inequality

$$|d_{21}(\gamma^{-i}x_2^+ - \mu + \gamma^{-j}x_2^+)| \leq S_{ij}(\bar{k}) \quad (23)$$

is necessary in order for the horse-shoe $T_{21}(\mu)(\sigma_j^{12})$ to have an irregular intersection with the strip σ_i^{01} .

Inequalities (21)-(23) have quite a simple geometrical interpretation, see Fig. 6: The strip σ_j^{12} is a thin rectangle with the central line $y_{12} = \gamma^{-j}x_2^+$. The strip σ_i^{01} is a thin rectangle with the central line $x_{01} = \gamma^{-i}x_2^+$. The strip σ_j^{01} is mapped by the map T_{21} onto a horse-shoe with the parabola $y_{01} = \mu - \gamma^{-j}x_2^+ +$

Figure 4:

Figure 5: Different types of intersection of strips and horseshoes

Figure 6:

Figure 7:

$d_{21}((x_{12} - x_2^+))^2$ as a central line. The condition $d_{21}(\gamma^{-i}x_2^+ - \mu + \gamma^{-j}x_2^+) > 0$ means that the straight line $y_{12} = \gamma^{-j}x_2^+$ and the parabola intersect in two points, and the condition $d_{21}(\gamma^{-i}x_2^+ - \mu + \gamma^{-j}x_2^+) < 0$ means that they have an empty intersection. The coefficient $S_{ij}(\bar{k})$ in (21)-(23) is due to the non-zero thicknesses of the strip and horse-shoe.

2.3 Dynamics near heteroclinic cycles

We consider the diffeomorphism f_0 , with initial non-transversal symmetric heteroclinic cycle.

It is well known that the properties of orbits in N essentially depend on the character of the intersection of strips and horseshoes. In the case when all strips and all horse-shoes intersect regularly, it can be proven [7, 9, 20] that all orbits from $N \setminus \Gamma_{21}$ are of saddle type. Moreover, N admits a full description in terms of symbolic dynamics: the orbits of N are in one-to-one correspondence with orbits of a subsystem of the topological Bernoulli shift on four symbols $\{1, 2, 3, 4\}$. Hence, an analysis of the structure of integer solutions of inequalities (21)-(23) is an essential part of the study of orbits of the set N . The sets of such solutions obviously depend on the signs of parameters λ , γ and d_{21} . Geometrically, this is related to the fact that the signs of these values determine the character of the reciprocal position of the manifolds $W_{\text{loc}}^s(O_1)$ and $T_1 W_{\text{loc}}^u(O_2)$ in a neighborhood of the heteroclinic point M_1^+ .

We now consider the cases of positive and negative multipliers, separately.

The case of positive λ and γ . The different possible cases here depend on the signs of d_{21} and types of reversibility are shown in Fig. 1. It follows from (22), when $d_{21} < 0$ and $\lambda > 0$ and $\gamma > 0$, that the horse-shoes $T_{21}\sigma_i^{12}$ and the strips σ_j^{01} do not intersect, since they lie at opposite sides of $W_{\text{loc}}^s(O_1)$, see Fig. 7a. It is obvious that in this case the set N has a trivial structure: $N = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$. In terms of the classification given by Gonchenko, Turaev & Shilnikov [7, 20], the diffeomorphisms under consideration are called *diffeomorphisms of the first class*.

In case $d_{21} > 0$, evidently, inequality (21) holds for sufficiently large i and j : the intersection of $T_{21}\sigma_i^{12}$ with σ_j^{01} is regular for any $i, j \geq \bar{k}$ (Fig. 7). In this case all orbits from $N \setminus \Gamma$ are of saddle

Figure 8: A priori non-symmetric heteroclinic cycles

type. The set N is a non-uniformly hyperbolic set and admits a full description in terms of symbolic dynamics [20]. According to [20] such diffeomorphisms belong to the *second class* of diffeomorphisms with a non-transversal heteroclinic cycle.

Remark 1 In [20] it has been shown that in general diffeomorphisms, there is also a *third class* of diffeomorphisms with non-transverse heteroclinic cycle, where the set N_0 , in general, does not admit a full description and contains a nontrivial hyperbolic subset. Moreover, on the bifurcation surface of these diffeomorphisms there is a dense set of diffeomorphisms with structurally unstable periodic and homoclinic orbits (with any order of degeneracy). For the case of positive multipliers of saddle fixed points, diffeomorphisms with the above-mentioned heteroclinic cycle can belong to the third class only if $c_{21} > 0$.

It is important to realize that we consider a symmetric heteroclinic cycle. We proved before that due to the reversibility, the coefficient c_{21} is always negative ($c_{21} = -1$). This implies that in the case of positive multipliers of saddle fixed points, there are no diffeomorphisms of the third class [20].

In turn, this means that a non-transversal heteroclinic orbit of f with the geometry of manifolds as sketched in Fig. 8 cannot be symmetric with respect to the reversing symmetry R . We say, in this case, that f has an a priori non-symmetric heteroclinic cycle.

The case of negative λ and γ . In this case f is a diffeomorphism of the *third class* (Fig. 7c, Fig. 7d). But there is a principal difference with the non-reversible case. Namely, it follows from the inequalities (21-23) (at $\mu = 0$) that the intersection of strips and horseshoes cannot be irregular, so that the set $N \setminus \Gamma_{21}$ contains only saddle type orbits. Again the set $N \setminus \Gamma_{21}$ is non-uniformly hyperbolic and is topologically conjugated to a subsystem of a Bernoulli shift. Of course, such type of diffeomorphisms are not typical for non-reversible diffeomorphisms.

3 Newhouse regions for homoclinic and heteroclinic tangencies

In this section we prove the first part of Theorem 1, concerning the density of diffeomorphisms with non-transversal homoclinic and heteroclinic cycles in parameter intervals. It is worth mentioning that conceptually, the proof is close to results in [20].

Figure 9:

Recall that in case $\lambda > 0$ and $\gamma > 0$, if $\mu \leq 0$ the diffeomorphism f_μ has trivial dynamics for $d_{21} < 0$, and hyperbolic behaviour for $d_{21} > 0$ (in this case the nonwandering set $N(\mu)$ admits full description in the terms of symbolic dynamics). Therefore, Newhouse regions can appear only for positive μ . In case $\lambda > 0, \gamma < 0$, generally speaking, dynamics of f_μ is non-uniformly hyperbolic only if $\mu = 0$ and Newhouse regions arise both when $\mu > 0$ and $\mu < 0$. Here, we will discuss in detail only the case $\mu > 0$, although the proofs can be carried through for negative values of μ .

From now on we assume that $\mu \geq 0$.

3.1 Density of diffeomorphisms with homoclinic tangencies

Lemma 2 *There exists a sequence $\{\mu_i\}$ of parameter values μ such that $\mu_i \rightarrow 0$ as $i \rightarrow \infty$, $\mu_i > 0$, and the diffeomorphism f_{μ_i} has a structurally unstable single-round orbit Γ_{1i} homoclinic to O_1 .¹, such that the tangency of the manifolds $W^s(O_1)$ and $W^u(O_2)$ along Γ_{1i} is quadratic and, for $\mu = \mu_i$, f_μ is a transversal family which unfolds the non-transversal homoclinic orbit Γ_{1i} generically.*

Proof

The idea is that we iterate a segment of unstable manifolds of O_1 to obtain a segment of the same manifolds in the neighbourhood Π_1^+ (Fig. 9). We denote as $l_u^1(\mu)$ the segment $T_{12}(W_{\text{loc}}^u(O_1)) \cap \Pi_2^+$ of the unstable manifold of the point O_1 . For small μ , the segment $l_u^1(\mu)$ intersects the manifold $W_{\text{loc}}^s(O_2)$ transversally. It is easy to see from (20) that the curves $l_u^{1i} \equiv T_{02}^i(l_u^1 \cap \sigma_i^{02})$ on Π_2^- accumulate to the segment $l_u^2 \equiv W_{\text{loc}}^u(O_2) \cap \Pi_2^-$ (Fig. 9), which is defined by the equation $x_{12} = 0$ on Π_2^- . In this case, by the virtue of (20) and (5), the equation of the curve l_u^{1i} is $x_{12} = \gamma(\mu)^{-i} x_2^+(1 + \dots)$ and, by virtue of (14), for $\mu = \mu_i \equiv \gamma^{-i} x_2^+(1 + \dots)$ the diffeomorphism f_μ has in U a single-round non-transversal homoclinic orbit Γ_{1i} of the point O_1 . For large i , the tangency of the stable and the unstable manifold of the point O_1 at the points of the orbit Γ_{1i} is quadratic, and the family f_μ is transversal to the bifurcation surface

¹For a definition of single-round orbits, see Section 4.

of diffeomorphisms with a non-transversal homoclinic orbit which is close to Γ_{1i} . ■

Remark 2 *It follows from reversibility that f_{μ_i} has a homoclinic tangency also at the point O_2 . Indeed, f_{μ_i} has a structurally unstable single-round homoclinic orbit Γ_{2i} which is the image of Γ_{1i} under R .*

Lemma 3 *There exists a sequence of intervals $\{\delta_i\}$ of values of the parameter μ which accumulate at $\mu = 0$ such that parameter values μ at which the family f_μ unfolds generically the homoclinic tangency of the point O_1 (and O_2) are dense in the intervals δ_i .*

Proof This lemma follows directly from the Newhouse theorem [26], lemma 2 and remark 2. ■

3.2 Density of diffeomorphisms with symmetric heteroclinic cycles

Since $\Gamma_{21} = R(\Gamma_{21})$, it is straightforward to prove that there exist a point q , $q \in \Gamma_{21}$, such that $q \in \text{Fix}(R) \cup \text{Fix}(R \circ f)$. By our assumptions, the sets $\text{Fix}(R)$ and $\text{Fix}(R \circ f)$ are one-dimensional curves. Moreover, it can be shown that the stable manifold W^s of O_1 and the unstable manifold W^u of O_2 have a quadratic tangency with the set $\text{Fix}(R)$ from opposite directions at the same point q . We recall that there exists a positive integer n_2 such that $f_0^{n_2}(M_2^-) = M_1^+$. If n_2 is even then $q \in \text{Fix}(R)$. For odd n_2 , the point q belongs to $\text{Fix}(R \circ f)$. We now suppose that $q \in \text{Fix}(R)$. (The proof is analogous if $q \in \text{Fix}(R \circ f)$.) Let $s_1(\mu)$ denote the segment $f_\mu^{n_2/2}(\text{Fix}(R)) \cap \Pi_1^+$. The curve $s_1(\mu)$ has a tangency with $W_{\text{loc}}^s(O_1)$ at the point M_2^- .

Proposition 1 *For any $\mu \in (0, \mu_0)$ and sufficiently large j (independent of μ) there exist constants l , $l = l(i, j)$ such that $f^l(\text{Fix}(R))$ accumulate to $W_{\text{loc}}^u(O_1)$ in Π_1^- and the distance between $f^l(\text{Fix}(R))$ and $W_{\text{loc}}^u(O_1)$ is approximately equal to $x_1^+ \lambda^j$.*

Proof Let us choose some μ inside the interval $(0, \mu_0)$. We discuss the cases $d_{21} < 0$ and $d_{21} > 0$ separately.

$d_{21} < 0$: In this case we have a simplest construction of the proof. Indeed, for $\mu = 0$ in neighborhood Π_1^+ there exist a tangency of $W_{\text{loc}}^s(O_1)$, $W_{\text{loc}}^u(O_2)$ and the curve $s_1(\mu)$ at the point M_1^+ (Fig. 10a). We can split this tangency. Let us choose the parameter μ from the interval $(0, \mu_0)$. We obtain two new transversal heteroclinic points M_{2a}^- and M_{2b}^- . The last belong to new symmetric heteroclinic orbits, therefore $s_1(\mu)$ must intersect $W_u(O_2)$ transversally in M_{2a}^- and M_{2b}^- also (Fig. 10b). As a consequence, for sufficiently large j ($j > \tilde{k}(\mu)$) the curve $s_1(\mu)$ intersects the strip σ_j^{01} along two components s_{j1}^{01} and s_{j2}^{01} (Fig. 10). By the virtue of (20), as $j \rightarrow \infty$, the curves $s_{j1}^{11} \equiv T_{01}^j(s_{j1}^{01})$ on Π_1^- accumulate to the segment $l_u^1 \equiv W_{\text{loc}}^u(O_1) \cap \Pi_1^-$, which is defined by the equation $x_{11} = 0$ on Π_1^- . It follows from (20) that the distance between s_{j1}^{11} and $W_{\text{loc}}^u(O_1)$ is approximately equal to $x_1^+ \lambda^j$. By construction, $s_{j1}^{11} = f^l(\text{Fix}(R)) \cap \Pi_1^-$, where $l = n_2/2 + j$.

$d_{21} > 0$: At $\mu = 0$, the curve $s_1(\mu)$ intersects the countable set of strips σ_i^{01} (Fig. 11a), where $i \geq \tilde{k} \geq \bar{k}$, $\tilde{k}$ is some integer. Hence, there exist μ_0 and $k = k(\mu_0)$ such that the curve $s_1(\mu)$ intersects σ_k^{01} along two components s_{k1}^{01} and s_{k2}^{01} for all $\mu \in (0, \mu_0)$, see Fig. 11b. It should be note that in this case the constant k depends on μ and we can not assume $j = k$ (as in the proof when $d_{21} < 0$). By virtue of (20), for large k (and small μ) the curves $s_{k1}^{11} \equiv T_{01}^k(s_{k1}^{01})$ on Π_1^- accumulate to the segment $l_u^1 \equiv W_{\text{loc}}^u(O_1) \cap \Pi_1^-$. It follows from (5) that in the neighbourhood Π_2^+ , $s_{k1}^{02} \equiv T_{12}(s_{k1}^{11})$ will accumulate to the segment $T_{12}(l_u^1) \cap \Pi_2^+$ of the unstable manifolds of O_1 . and hence will intersect the strips σ_i^{02} at i , where $i \leq \bar{k}$ ($\bar{k}$ not dependent on μ). From (18) we obtain that the family $s_{ki1}^{12} \equiv T_{02}^i(s_{k1}^{02} \cap \sigma_i^{02})$ on Π_2^- belongs to the strips σ_i^{12} at all $i \leq \bar{k}$. By virtue of (18), the equation of s_{ki1}^{12} is $x_{12} = \gamma(\mu)^{-i} x_2^+(\mu)(1 + \dots)$.

It follows from (14) that for $\mu < \gamma(\mu)^{-i} x_2^+$ the image $T_{21}(s_{ki1}^{12})$ has two transversal intersection with the stable manifold $W_{\text{loc}}^s(O_1)$. Hence, for sufficiently large j , $s_1(\mu)$ intersects the strip σ_j^{01} along two

a) $\mu=0$

b) $\mu > 0$

Figure 10:

a) $\mu=0$

b) $\mu > 0$

Figure 11:

components s_{kij1}^{01} and s_{kij2}^{01} , see Fig. 11b. By virtue of (20), as $j \rightarrow \infty$, the curves $s_{kij1}^{11} \equiv T_{01}^j(s_{kij1}^{01})$ on Π_1^- accumulate to the segment $l_u^1 \equiv W_{\text{loc}}^u(O_1) \cap \Pi_1^-$. From (20) we can calculate the distance between s_{j1}^{11} and $W_{\text{loc}}^u(O_1)$ which is approximately equal to $x_1^+ \lambda^j$. By construction, $s_{kij1}^{11} = f^l(\text{Fix}(R)) \cap \Pi_1^-$, where $l = n_2/2 + k + n_1 + i + n_2 + j$. ■

Lemma 4 *In the intervals δ_i , the set of parameter values μ for which the family f_μ unfolds generically the symmetric type heteroclinic cycle, consisting of two saddle fixed points O_1 and O_2 , heteroclinic orbits Γ_{12} and new non-transversal heteroclinic orbits Γ_{21}^{l1} , is dense.*

Proof It follows from lemma 3 that in each of the intervals δ_i there is a dense set $\tilde{\delta}_i$ such that for all $\mu_i^* \in \tilde{\delta}_i$ the diffeomorphism $f_{\mu_i^*}$ has a homoclinic tangency with respect to fixed point O_1 . According to Proposition 1 we can split this homoclinic tangency in such way that the diffeomorphism $f_{\mu_{ij}^*}$ will have a tangency of the unstable manifold of O_1 and the curve $f^l(\text{Fix}(R)) \cap \Pi_1^-$, where $\mu_{ij}^* - \mu_i^*$ will be approximately equal to $x_1^+ \lambda^j$, see the proof of Proposition 1. By reversibility, each such a tangency corresponds to the existence of a non-transversal heteroclinic cycle containing the points O_1 and O_2 . Finally, by choosing j sufficiently large, we prove the density of parameter values μ_{ij}^* in the interval $\delta_i(\mu)$. ■

Remark 3 *It can be shown that Newhouse intervals δ_i in fact contain dense sets of diffeomorphisms having non-transversal symmetric heteroclinic cycles with tangencies of all possible types (see the classification of tangency with respect to the reciprocal position of the manifolds in Section 2.3).*

In order to obtain such other types of tangency, we repeat the construction of proposition 1 but we choose the component s_{j2}^{01} instead s_{j1}^{01} in the case $d_{21} < 0$ and the component s_{kij2}^{01} instead s_{kij1}^{01} in the case $d_{21} > 0$.

3.3 Density of diffeomorphisms with non-symmetric heteroclinic cycles

Lemma 5 *The set of parameter values μ^* , such that f_{μ^*} has a nonsymmetric heteroclinic cycle $C = \{O_1, O_2, \Gamma_{12}, \tilde{\Gamma}_{21}\}$, and f_μ unfolds the non-transversal heteroclinic orbits $\tilde{\Gamma}_{21}$ generically, is dense in δ_i .*

Proof

Let us choose the parameter μ^* such that the diffeomorphisms f_{μ^*} has the heteroclinic cycle which is depicted in Fig. 7b and Fig. 7d. In general, such diffeomorphisms are selected by the conditions $d_{21} > 0$ and the sign of multipliers can be also negative. We claim that under small perturbation we can obtain a diffeomorphism with an "a priori non-symmetric heteroclinic" cycle (Fig. 8). For the diffeomorphisms with positive multipliers of saddle fixed points such heteroclinic cycle belongs to the third class. It is clear that in this case f_{μ^*} has a non-transversal heteroclinic point M_{1ijk}^- which is close to M_1^- . Of course, the unstable manifold W^u of the point O_2 intersects the strip σ_m^{01} along two components ω_{m1}^{01} and ω_{m2}^{01} (Fig. 12) for sufficiently large m . Let us iterate ω_{m1}^{01} :

$$\omega_{m1}^{11} = T_{01}^m(\omega_{m1}^{01}); \quad \omega_{ml1}^{02} = T_{12}(\omega_{m1}^{11}) \cap \sigma_l^{02};$$

$$\omega_{ml1}^{12} = T_{02}^l(\omega_{ml1}^{02}); \quad \omega_{ml1}^{02} = T_{21}(\omega_{ml1}^{12}).$$

The process of the iterations is illustrated in Fig. 12. It is easy to see that the curves ω_{ml1}^{02} have the shape of a "parabola". By a small perturbation of μ (corresponding to a sufficiently large l , as $\mu - \mu^* = O(\gamma^{-l})$) we can obtain a tangency of the "parabola" ω_{ml1}^{02} (for a certain $l \geq \bar{k}$) and the segment of the stable manifold $W_{\text{loc}}^s(O_1) \cap \Pi_1^+$ of O_1 . It is readily verified that this tangency is associated with an "a priori non-symmetric heteroclinic cycle" ■

This completes the proof of parts 1 – 3 of Theorem 1.

Figure 12:

4 Bifurcations of single-round periodic orbits

In this section we study bifurcations of *single-round periodic orbits*, lying entirely in U . Such orbits have exactly one intersection point with Π_l^+ and Π_l^- , $l = 1, 2$. Let Λ be some single-round periodic orbit and let P_l^+ and P_l^- be its intersection points with Π_l^+ and Π_l^- , respectively. Then, there exist some integers $i \geq \bar{k}$ and $j \geq \bar{k}$ such that

$$P_1^+ \in \sigma_i^{01}, P_1^- \in \sigma_i^{11}, P_2^+ \in \sigma_j^{02}, P_2^- \in \sigma_j^{12}.$$

Moreover, the following relations hold:

$$P_1^- = T_{01}^i(P_1^+), P_2^+ = T_{12}(P_1^-), P_2^- = T_{02}^j(P_2^+), P_1^+ = T_{21}(P_2^-).$$

Thus, the point P_1^+ can be considered as a fixed point of a first return map $T_{ij}(\mu)$ defined as the following composition of the local and global maps

$$T_{ij} \equiv T_{21} \cdot T_{02}^j \cdot T_{12} \cdot T_{01}^i : \sigma_i^{01} \mapsto \Pi_1^+. \quad (24)$$

The study of bifurcations of single-round periodic orbits in the family f_μ is reduced to the study of bifurcations of fixed points of the first return maps $T_{ij}(\mu)$ for various sufficiently large i and j . Formulas (5), (14) and (18) make it possible to find an explicit expression for the map $T_{ij}(\mu)$. In fact, it turns out that by appropriate rescaling, $T_{ij}(\mu)$ can be put into a convenient standard form. This is the topic of the next sections.

4.1 Construction of the Poincaré map T_{ii} and Rescaling Lemma.

In this section, we construct the first-return maps that describe symmetric periodic orbits. In order for a fixed point of T_{ij} to be able to be a symmetric periodic point, it is easily verified that i must be equal

to j . Hence we first focus on T_{ii} . We have

$$T_{ii} = T_{21} \circ T_{02}^i \circ T_{12} \circ T_{01}^i : \sigma_i^{01} \rightarrow \Pi_k^+.$$

It follows from (6), (15) and (19) that

$$\begin{aligned} T_{ii}^{-1} &= T_{01}^{-i} \circ T_{12}^{-1} \circ T_{02}^{-i} \circ T_{21}^{-1} = \\ R_{12}^{-1} \circ T_{02}^i \circ R_{12} \circ R_{12}^{-1} \circ T_{12} \circ R_{12}^{-1} \circ R_{12} \circ T_{01}^i \circ R_{12}^{-1} \circ R_{12} \circ T_{21} R_{12} &= \\ R_{12}^{-1} \circ T_{02}^i \circ T_{12} \circ T_{01}^i \circ T_{21} R_{12}. \end{aligned}$$

This means that there exists an

$$R_* = T_{01}^{-i} \circ T_{12}^{-1} \circ T_{02}^{-i} \circ R_{12},$$

such that

$$T_{ii}^{-1} = R_* \circ T_{ii} \circ (R_*)^{-1}.$$

Note that $R_* = (R^*)^{-1}$, and in general R^* is nonlinear. But, special coordinates can be chosen such that in terms of these coordinates, R^* is linear. In order to show this, we write the local and global maps, using a simplified representation of these maps, introducing functions p , q , Φ , F and G that represent the right-hand sides of the first and second equations of (18), (12) and (14), respectively:

$$\begin{aligned} T_{01}^i ((x_{01}, y_{01}) \rightarrow (x_{11}, y_{11})) : \quad & \begin{cases} x_{11} = p(x_{01}, y_{11}), \\ y_{01} = q(x_{01}, y_{11}); \end{cases} \\ T_{12} ((\bar{x}_{11}, \bar{y}_{11}) \rightarrow (\bar{x}_{02}, \bar{y}_{02})) : \quad & \begin{cases} \bar{x}_{02} - x_2^+ = \Phi(x_{11}, \bar{y}_{02}), \\ y_{11} - x_2^+ = \Phi(\bar{y}_{02}, x_{11}) \end{cases} \\ T_{02}^i ((\bar{x}_{02}, \bar{y}_{02}) \rightarrow (\bar{x}_{12}, \bar{y}_{12})) : \quad & \begin{cases} \bar{y}_{02} = p(\bar{y}_{12}, \bar{x}_{02}), \\ \bar{x}_{12} = q(\bar{y}_{12}, \bar{x}_{02}); \end{cases} \\ T_{21} ((\bar{x}_{12}, \bar{y}_{12}) \rightarrow (\bar{x}_{01}, \bar{y}_{01})) : \quad & \begin{cases} x_{01} - x_1^+ = F(x_{12}, y_{12} - x_1^+, \mu), \\ y_{01} = G(x_{12}, y_{12} - x_1^+, \mu). \end{cases} \end{aligned} \tag{25}$$

The map T_{01}^i is defined on the strip $\sigma_i^{01} = \Pi_1^+ \cap T_{01}^{-i}(\Pi_1^-)$. Hence, without loss of generality we can choose the new coordinates (ξ_1, η_1) on σ_i^{01} , where $\xi_1 = x_{01} - x_1^+$, $\eta_1 = y_{11} - x_2^+$. It is easy to see that subsequently

$$\begin{aligned} \bar{x}_{01} &= x_1^+ + \xi_1, \\ \bar{y}_{01} &= q(x_1^+ + \xi_1, x_2^+ + \eta). \end{aligned}$$

Analogously, on the strip $\sigma_i^{02} = \Pi_2^+ \cap T_{02}^{-i}(\Pi_2^-)$ we choose coordinates $(\bar{\xi}_2, \bar{\eta}_2)$, where $\bar{\xi}_2 = x_{02} - x_2^+$, $\bar{\eta}_2 = y_{12} - x_1^+$, so that

$$\begin{aligned} \bar{x}_{02} &= x_2^+ + \bar{\xi}_2, \\ \bar{y}_{02} &= p(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2). \end{aligned}$$

Rewriting the maps T_{01}^i , T_{12} , T_{02}^i and T_{21} in terms of these coordinates, we obtain

$$\begin{aligned} T_{01}^i ((\xi_1, \eta_1) \rightarrow (x_{11}, y_{11})) : \quad & \begin{cases} x_{11} = p(x_1^+ + \xi_1, x_2^+ + \eta), \\ y_{11} = x_2^+ + \eta; \end{cases} \\ T_{12} ((\bar{x}_{11}, \bar{y}_{11}) \rightarrow (\bar{\xi}_2, \bar{\eta}_2)) : \quad & \begin{cases} \bar{\xi}_2 = \Phi(x_{11}, p(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2)), \\ y_{11} - x_2^+ = \Phi(p(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), x_{11}) \end{cases} \\ T_{02}^i ((\bar{x}_{02}, \bar{y}_{02}) \rightarrow (\bar{x}_{12}, \bar{y}_{12})) : \quad & \begin{cases} \bar{x}_{12} = q(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), \\ \bar{y}_{12} = x_1^+ + \bar{\eta}_2; \end{cases} \\ T_{21} ((\bar{x}_{12}, \bar{y}_{12}) \rightarrow (\bar{\xi}_1, \bar{\eta}_1)) : \quad & \begin{cases} \bar{\xi}_1 = F(\bar{x}_{12}, \bar{y}_{12} - x_1^+, \mu), \\ q(x_1^+ + \bar{\xi}_1, x_2^+ + \bar{\eta}_1) = G(\bar{x}_{12}, \bar{y}_{12} - x_1^+, \mu). \end{cases} \end{aligned} \tag{26}$$

Therefore, the map $T_{12}T_{01}^i : (\xi_1, \eta_1) \rightarrow (\bar{\xi}_2, \bar{\eta}_2)$ can be written in the form

$$\begin{aligned}\bar{\xi}_2 &= \Phi(p(x_1^+ + \xi_1, x_2^+ + \eta_1), p(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2)), \\ \eta_1 &= \Phi(p(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), p(x_1^+ + \xi_1, x_2^+ + \eta_1))\end{aligned}\quad (27)$$

It is easy to see that the last equations define $\bar{\xi}_2$ and η_1 uniquely as a function of ξ_1 and $\bar{\eta}_2$. Indeed, we can resolve this system with respect to $(\xi_1, \bar{\eta}_2)$, obtaining

$$\begin{aligned}\bar{\xi}_2 &= \Psi(\xi_1, \bar{\eta}_2), \\ \eta_1 &= \Psi(\bar{\eta}_2, \xi_1).\end{aligned}\quad (28)$$

It follows from (26) that the map $T_{21}T_{02}^i : (\bar{\xi}_2, \bar{\eta}_2) \rightarrow (\bar{\bar{\xi}}_1, \bar{\bar{\eta}}_1)$ has the following form

$$\begin{aligned}\bar{\bar{\xi}}_1 &= F(q(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), \bar{\eta}_2, \mu), \\ q(x_1^+ + \bar{\bar{\xi}}_1, x_2^+ + \bar{\bar{\eta}}_1) &= G(q(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), \bar{\eta}_2, \mu).\end{aligned}\quad (29)$$

Now, we choose the new coordinates

$$\xi = \xi_1, \quad \eta = \bar{\eta}_2$$

on the strip σ_i^{01} , so that the maps $T_{12}T_{01}^i : (\xi, \eta) \rightarrow (\bar{\xi}_2, \bar{\eta}_2)$ and $T_{21}T_{02}^i : (\bar{\xi}_2, \bar{\eta}_2) \rightarrow (\bar{\bar{\xi}}_1, \bar{\bar{\eta}}_1)$ can be rewritten as

$$\begin{aligned}\bar{\xi}_2 &= \Psi(\xi, \eta), \\ \bar{\eta}_2 &= \eta\end{aligned}\quad (30)$$

and

$$\begin{aligned}\bar{\bar{\xi}}_1 &= F(q(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), \bar{\eta}_2, \mu), \\ q(x_1^+ + \bar{\bar{\xi}}_1, x_2^+ + \bar{\bar{\eta}}_1) &= G(q(x_1^+ + \bar{\eta}_2, x_2^+ + \bar{\xi}_2), \bar{\eta}_2, \mu),\end{aligned}\quad (31)$$

respectively.

Let us substitute the values of $\bar{\xi}_2$ and $\bar{\eta}_2$ given by (30) into (31). Then we find that the map T_{ii} has the form

$$\begin{aligned}\bar{\bar{\xi}}_1 &= F(q(x_1^+ + \eta, x_2^+ + \Psi(\xi, \eta)), \eta, \mu), \\ q(x_1^+ + \bar{\bar{\xi}}_1, x_2^+ + \bar{\bar{\eta}}_1) &= G(q(x_1^+ + \eta, x_2^+ + \Psi(\xi, \eta)), \eta, \mu).\end{aligned}\quad (32)$$

It follows from the reversibility properties of T_{21} (see (6),(7),(8)), that the functions F and G satisfy:

$$\text{if } \begin{cases} z_1^* = F(z_1, z_2) \\ z_2^* = G(z_1, z_2) \end{cases} \text{ then } \begin{cases} z_2 = F(z_2^*, z_1^*) \\ z_1 = G(z_2^*, z_1^*) \end{cases}$$

Hence, from (32) we obtain

$$\begin{aligned}\eta &= F(q(x_1^+ + \bar{\bar{\xi}}_1, x_2^+ + \bar{\bar{\eta}}_1), \bar{\bar{\xi}}_1, \mu), \\ q(x_1^+ + \eta, x_2^+ + \Psi(\xi, \eta)) &= G(q(x_1^+ + \bar{\bar{\xi}}_1, x_2^+ + \bar{\bar{\eta}}_1), \bar{\bar{\xi}}_1, \mu).\end{aligned}\quad (33)$$

The equations (32) and (33) provide an implicit form of the maps T_{ii} and T_{ii}^{-1} , respectively. From these expressions, one readily verifies the following result:

Proposition 2 *In coordinates (ξ, η) the first-return map T_{ii} is reversible with respect to the linear involution $\tilde{R}$, where $\tilde{R}(\xi, \eta) = (\eta, \xi)$. That is,*

$$T_{ii} = \tilde{R} \circ T_{ii}^{-1} \circ \tilde{R}.$$

After normalizing the reversing symmetry, we now seek to normalize the map T_{ii} further by a (rescaling) coordinate transformation. In order to preserve the $\tilde{R}$ -reversibility, we require such a coordinate transformation to be $\tilde{R}$ -equivariant. With $\tilde{R}$ as defined above, this implies that we allow only coordinate transformations of the form $(\xi, \eta) \rightarrow (X, Y)$, where $(X, Y) = (h(\xi, \eta), h(\eta, \xi))$. We obtain the following:

Lemma 6 (Rescaling Lemma) *In some rescaled coordinates (X, Y) the map $T_{ii}(\mu)$ is $\tilde{R}$ -reversible and can be written in the following form*

$$\begin{aligned}\bar{X} &= Y + \varepsilon_{1i}, \\ \bar{Y} &= M - X - Y^2 + \varepsilon_{2i}\end{aligned}\tag{34}$$

where

$$M = -d_{21}/A^2 [\mu - 2x_2^+ \gamma^{-i} + \dots] \gamma^{2i} \lambda^{-2i},\tag{35}$$

and ε_{li} , $l = 1, 2$, denote some functions such that

$$\|\varepsilon_{li}(X, Y, M)\|_{C^{r-2}} = O(\gamma^{-i}).$$

In the rescaled coordinates (X, Y) , the domain where the map T_{ii} is defined includes the disc $\|X, Y\| \leq K_{ii}$ where $K_{ii} \rightarrow +\infty$ as $i \rightarrow +\infty$.

Proof By Proposition 2, we know that the map T_{ii} can be written in coordinates (ξ, η) , so that T_{ii} is $\tilde{R}$ -reversible. We first examine more closely this procedure, after which we carry out the rescaling procedures.

As discussed above, we choose the coordinates (ξ_1, η_1) on σ_i^{01} , where $\xi_1 = x_{01} - x_1^+$, $\eta_1 = y_{11} - x_2^+$ and the coordinates $(\bar{\xi}_2, \bar{\eta}_2)$ on σ_i^{02} , where $\bar{\xi}_2 = x_{02} - x_2^+$, $\bar{\eta}_2 = y_{12} - x_1^+$.

The map $T_{21}T_{02}^i : (\bar{\xi}_2, \bar{\eta}_2) \rightarrow (\bar{\xi}_1, \bar{\eta}_1)$ is implicitly given by (27). It follows from (13), (18), and (26), that

$$p(u, v) = \lambda(\mu)^i u (1 + \gamma^{-i} p_i(u, v, \mu))$$

and

$$\Phi(u, v, \mu) = A(\mu)u + B(\mu)v + O(u^2 + |uv| + v^2).$$

Substituting these expressions for the functions p and Φ into (27), we obtain

$$\begin{aligned}\bar{\xi}_2 &= A\lambda^i x_1^+ (1 + \dots) + B\lambda^i x_1^+ (1 + \dots) + A\lambda^i \xi_1 (1 + \dots) + \\ &\quad B\lambda^i \bar{\eta}_2 (1 + \dots) + O(|\lambda^i \gamma^{-i}| (|\eta_1| + |\bar{\xi}_2| + \xi_1^2 + \bar{\eta}_2^2 + |\xi_1 \bar{\eta}_2|))\end{aligned}\tag{36}$$

$$\begin{aligned}\eta_1 &= A\lambda^i x_1^+ (1 + \dots) + B\lambda^i x_1^+ (1 + \dots) + A\lambda^i \bar{\eta}_2 (1 + \dots) + \\ &\quad B\lambda^i \xi_1 (1 + \dots) + O(|\lambda^i \gamma^{-i}| (|\bar{\xi}_2| + |\eta_1| + \bar{\eta}_2^2 + \xi_1^2 + |\bar{\eta}_2 \xi_1|))\end{aligned}$$

where the dots stand for the constant terms (i.e. independent of ξ, η) which tend to zero as $i \rightarrow \infty$, and have asymptotical order γ^{-i} .

It is easy to see that it is possible to resolve $\bar{\xi}_2$ and η_1 from (36), giving

$$\begin{aligned}\bar{\xi}_2 &= A\lambda^i x_1^+ (1 + \dots) + B\lambda^i x_1^+ (1 + \dots) + A\lambda^i \xi_1 (1 + \dots) + \\ &\quad B\lambda^i \bar{\eta}_2 (1 + \dots) + O(|\lambda^i \gamma^{-i}| (\xi_1^2 + \bar{\eta}_2^2 + |\xi_1 \bar{\eta}_2|)) \\ \eta_1 &= A\lambda^i x_1^+ (1 + \dots) + B\lambda^i x_1^+ (1 + \dots) + A\lambda^i \bar{\eta}_2 (1 + \dots) + \\ &\quad B\lambda^i \xi_1 (1 + \dots) + O(|\lambda^i \gamma^{-i}| (\bar{\eta}_2^2 + \xi_1^2 + |\bar{\eta}_2 \xi_1|))\end{aligned}\tag{37}$$

Now we consider the map $T_{21}T_{02}^i : (\bar{\xi}_2, \bar{\eta}_2) \rightarrow (\bar{\bar{\xi}}_1, \bar{\bar{\eta}}_1)$. Schematically, it is represented by the system of equations (29). From (14), (18), and (26), we have expressions for the functions F , G and q :

$$F(u, v) = -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}(\mu)u + (1 + b_\mu\mu)v + O(u^2 + |uv| + v^2),$$

$$G(u, v) = \mu - (1 - c_\mu\mu)u + d_{21}(\mu)v^2 + O(u^2 + |uv| + v^3)$$

and

$$q(u, v) = \gamma(\mu)^{-i}v(1 + \gamma^{-i}q_i(u, v, \mu)).$$

Substituting these expressions into (29), we obtain that $T_{21}T_{02}^i$ takes the form

$$\begin{aligned} \bar{\bar{\xi}}_1 &= -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}\gamma^{-i}x_2^+(1 + \dots) + a_{21}\gamma^{-i}\bar{\xi}_2(1 + \dots) \\ &\quad + (1 + b_\mu\mu)\bar{\eta}_2 + O(\gamma^{-2i}\bar{\xi}_2^2 + \gamma^{-2i}|\bar{\xi}_2\bar{\eta}_2| + \bar{\eta}_2^2), \\ \gamma^{-i}x_2^+(1 + \dots) + \gamma^{-i}\bar{\bar{\eta}}_1(1 + \dots) &\left[1 + O\left(|\gamma^{-i}|(|\bar{\eta}_1| + |\bar{\xi}_1|)\right) \right] + O(\gamma^{-2i}\bar{\bar{\xi}}_1) \\ &= \mu - (1 - c_\mu\mu)\gamma^{-i}x_2^+(1 + \dots) - (1 - c_\mu\mu)\gamma^{-i}\bar{\xi}_1(1 + \dots) + \\ &\quad O(\gamma^{-2i}\bar{\eta}_2) + d_{21}\bar{\eta}_2^2(1 + \dots) + O(\gamma^{-2i}\bar{\xi}_2^2 + \gamma^{-2i}|\bar{\xi}_2\bar{\eta}_2| + |\bar{\eta}_2|^3). \end{aligned} \quad (38)$$

Like before, we choose the new coordinates

$$\xi = \xi_1, \quad \eta = \bar{\eta}_2$$

on the strip σ_i^{01} and construct the composition of $T_{12}T_{01}^i$ and $T_{21}T_{02}^i$. We obtain

$$\begin{aligned} \bar{\bar{\xi}} &= -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}\gamma^{-i}x_2^+(1 + \dots) + a_{21}\gamma^{-i}\lambda^i\xi(1 + \dots) \\ &\quad + (1 + b_\mu\mu + \dots)\eta + O(|\lambda|^i\gamma^{-2i}\xi + |\lambda^i\gamma^{-2i}\xi\eta| + \eta^2), \\ \gamma^{-i}x_2^+(1 + \dots) + \gamma^{-i}\lambda^iA\bar{\bar{\eta}}(1 + \dots) &\left[1 + O\left(|\gamma^{-i}|(|\bar{\eta}| + |\bar{\xi}|)\right) \right] + \underline{O(\gamma^{-2i}\bar{\bar{\xi}})} \\ &= \mu - (1 - c_\mu\mu)\gamma^{-i}x_2^+(1 + \dots) - (1 - c_\mu\mu)\gamma^{-i}\lambda^i\xi(1 + \dots) + \\ &\quad \underline{O(\gamma^{-2i}\eta)} + d_{21}\eta^2(1 + \dots) + O(\gamma^{-2i}\xi^2 + \gamma^{-2i}|\bar{\xi}\eta| + |\eta|^3). \end{aligned} \quad (39)$$

Subsequently, we eliminate the underlined terms in the second equation of (39), using the following construction: we substitute the expression for $\bar{\bar{\xi}}$ which is given by the first equation of (39) into the left-hand side of the second equation of (39). If we differentiate both sides of the obtained second equation of (39) with respect to η , then - since $d_{21} \neq 0$ - the resulting equation defines η uniquely as a smooth function of ξ and $\bar{\bar{\eta}}$:

$$\eta = \Theta(\xi, \bar{\bar{\eta}}) = O(\gamma^{-2i}).$$

By construction, the following shift of the origin of the coordinates

$$\xi^{new} = \xi - \xi^*, \quad \eta^{new} = \eta - \eta^*,$$

where $\eta^* = \Theta(\xi^*, \eta^*)$, eliminates the underlined terms in (39). In order to preserve reversibility we have to choose the shift such that $\xi^* = \eta^*$. It turns out that we do so. Indeed, let us consider the system of equations

$$\Theta(\xi, \eta) = \eta, \quad \xi = \eta.$$

It is easy to see that it has a unique solution ($\xi = \eta^*, \eta = \eta^*$) where $\eta^* = O(\gamma^{-2i})$.

By the shift

$$\xi^{new} = \xi - \eta^*, \quad \eta^{new} = \eta - \eta^*,$$

we bring the system (39) into the form

$$\begin{aligned} \bar{\xi} &= -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}\gamma^{-i}x_2^+(1 + \dots) + a_{21}\gamma^{-i}\lambda^i\xi(1 + \dots) \\ &\quad + (1 + b_\mu\mu + \dots)\eta + O(|\lambda|^i\gamma^{-2i}\xi + |\lambda^i\gamma^{-2i}\xi\eta| + \eta^2) \\ \gamma^{-i}\lambda^i A \bar{\eta}(1 + \dots) &\left(1 + O\left(|\gamma^{-i}|(|\bar{\eta}| + |\bar{\xi}|)\right)\right) = \\ &[\mu - (2 + O(\mu) + \dots)\gamma^{-i}x_2^+] - (1 - c_\mu\mu)\gamma^{-i}\lambda^i\xi(1 + \dots) + \\ &d_{21}\eta^2(1 + \dots) + O(\gamma^{-2i}\xi^2 + |\gamma^{-2i}\bar{\xi}\eta| + |\eta|^3) \end{aligned} \quad (40)$$

The obtained map is also reversible because such type of coordinate shift does not destroy reversibility.

Note that all the constant (zero order) terms in the second equation of (40) are collected in the square brackets. For $\mu \approx 2\gamma^{-i}x_2^+(1 + \dots)$ it can be sufficiently small. Suppose that $M_i = [\mu - (2 + O(\mu) + \dots)\gamma^{-i}x_2^+] = O(\lambda^{2i}\gamma^{-2i})$. Consider the zero order terms M_{1i} in the first equation of (40). Since $\mu = (2 + \dots)\gamma^{-i}x_2^+$, then $M_{1i} = -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}\gamma^{-i}x_2^+(1 + \dots)$ has indefinite order of smallness. In order to estimate M_{1i} we use the property of reversibility. It is easy to see that T_{ii} transfer the point $(\xi = 0, \eta = 0)$ to (ξ^*, η^*) , where $\xi^* = M_{1i}$, $\eta^* = \frac{M_i}{A\gamma^{-i}\lambda^i}(1 + \dots) = O(\gamma^{-i}\lambda^i)$. According to reversibility, the point (η^*, ξ^*) should be transferred to the point $(0, 0)$. Due to (40), we obtain that

$$0 = M_{1i} + a_{21}\frac{M_i}{A}(1 + \dots) + M_{1i} + \dots$$

Hence, $M_{1i} = -a_{21}\frac{M_i}{2A}(1 + \dots)$. We obtain that M_{1i} has the same order of smallness.

By a further rescaling of variables

$$\begin{aligned} \xi &= -A/d_{21}\lambda^i\gamma^{-i}X \\ \eta &= -A/d_{21}\lambda^i\gamma^{-i}Y \end{aligned}$$

we bring system (40) to the form

$$\begin{aligned} \bar{X} &= Y + O(\gamma^{-i}) \\ \bar{Y}(1 + O(\gamma^{-i})) &= M - X - Y^2 + O(\gamma^{-i}) \end{aligned} \quad (41)$$

where

$$M = -(\mu - 2\gamma^{-i}x_2^+ + \dots)d_{21}/A^2\lambda^{-2i}\gamma^{2i}. \quad (42)$$

Now we can resolve the last system with respect to variables $\bar{X}$ and $\bar{Y}$. We obtain (34). The lemma is proved. ■

4.2 Bifurcations of the reversible Henon map

The first-return map T_{ii} is close to the reversible Henon map of the form

$$\begin{aligned}\bar{X} &= Y \\ \bar{Y} &= M - X - Y^2\end{aligned}\tag{43}$$

whose bifurcations have been well studied, so that it is possible to recover the bifurcation picture for the initial map T_{ii} . Indeed, bifurcations of single-round periodic orbits in the family f_μ , or -equivalently - bifurcations of fixed points of maps T_{ii} with sufficiently large i , can be studied if to know the structure of bifurcations of fixed points of map (43). It is easy to verify that the map (43) has a fixed point with double multipliers $+1$ at $M = -1$ and double multipliers -1 at $M = 3$. This map has no fixed points if $M < -1$. Moreover, we have an elliptic fixed point at $M \in (-1, 3)$. At the parameter value $M = M(\psi) = \cos^2\psi - 2\cos\psi$ the fixed point has multipliers $\nu_{1,2} = e^{\pm i\psi}$. The fixed point is a *generic* (Liapunov stable) elliptic point if $\psi \notin \{\pi/2, \arccos(-1/4), \pi/3\}$ (cf [1]). The map has an generic elliptic fixed point O_i at $-1 + \dots < M < 3 + \dots$, where $M \notin \{-3/4, 0, 9/16\}$. By the rescaling lemma, similar properties hold for the map T_{ii} . Using (42), it then follows that

Proposition 3 *In the parameter space there exist infinite many intervals $\delta(\mu)_i^{el}$ with boundary points*

$$\begin{aligned}\mu_i^{+1} &= 2\gamma^{-i}x_2^+(1 + \dots) + \frac{A^2}{d_{21}^2}\gamma^{-2i}\lambda^{2i} \\ \mu_i^{-1} &= 2\gamma^{-i}x_2^+(1 + \dots) - \frac{3A^2}{d_{21}^2}\gamma^{-2i}\lambda^{2i}\end{aligned}\tag{44}$$

accumulating to $\mu = 0$. For values of parameter μ from these intervals, the diffeomorphism f_μ has elliptic periodic points.

4.3 Bifurcations of non-symmetric periodic orbits. Rescaling lemma for the map T_{ij}

Now we consider non-symmetric single-round periodic orbits. Such orbits stay much more time near one saddle point than near the other. Recall that the study of bifurcations of single-round periodic orbits in the family f_μ may be reduced to the study of bifurcations of fixed points of the first return maps $T_{ij}(\mu)$, $T_{ij}(\mu) = T_{21}T_{02}^jT_{12}T_{01}^i$, for various sufficiently large i and j . In order to study non-symmetric periodic orbits we should suppose that $i \neq j$. In particular, we will focus on the case $i > j$. From (12), (14), and (18), we can find the explicit form of the map $T_{ij}(\mu)$ in the initial coordinates. We can rescale these coordinates and parameters in order to put the map $T_{ij}(\mu)$ in some standard form. The following rescaling lemma holds.

Lemma 7 *Let $i > j$. In some rescaled coordinates (X, Y) the map $T_{ij}(\mu_1, \mu_2)$ can be written in the following form*

$$\begin{aligned}\bar{X} &= Y, \\ \bar{Y} &= M - Y^2 + O((\lambda\gamma)^{(i-j)}),\end{aligned}\tag{45}$$

where

$$M = -\frac{d_{21}}{A^2} [\mu - x_2^+\gamma^{-i} + \dots] \gamma^{2i}\lambda^{-2j},\tag{46}$$

Here, the dots stand for terms which tend to zero as $i, j \rightarrow \infty$ (and which are independent of (X, Y)). In the rescaled coordinates (X, Y) , the domain where the map T_{ij} is defined includes the disc $\|X, Y\| \leq K_{ij}$ where $K_{ij} \rightarrow +\infty$ as $i, j \rightarrow +\infty$.

This lemma is a particular case of the rescaling lemma in [12]². For selfcontainment we provide a proof, simplifying the exposition of [12].

²Note that in this lemma we do not require to preserve reversibility.

Proof Like we did before, we will use (ξ_1, η_1) and (ξ_2, η_2) as the coordinates on the strips σ_i^{01} and σ_j^{02} respectively, where

$$\begin{aligned} x_{01} - x_1^+ &= \xi_1 & x_{02} - x_2^+ &= \xi_2, \\ y_{11} - x_2^+ &= \eta_1 & y_{12} - x_1^+ &= \eta_2. \end{aligned} \quad (47)$$

Then, by virtue of (18) and (12), the map $T_{12}T_{01}^i : (\xi_1, \eta_1) \mapsto (\bar{\xi}_2, \bar{\eta}_2)$ is written as

$$\begin{aligned} \bar{\xi}_2 &= A\lambda^i x_1^+ (1 + \dots) + A\lambda^i (1 + \dots)\xi_1 + B\lambda^j x_1^+ (1 + \dots)\bar{\eta}_2 + \\ &\quad O(\lambda^i \gamma^{-i} (|\eta_1| + |\bar{\xi}_2| + \xi_1^2 + \bar{\eta}_2^2 + |\xi_1 \bar{\eta}_2|)), \\ \eta_1 &= A\lambda^j x_1^+ (1 + \dots) + A\lambda^j (1 + \dots)\bar{\eta}_2 + B\lambda^i x_1^+ (1 + \dots) \\ &\quad + B\lambda^i (1 + \dots)\xi_1 + O(\lambda^i \gamma^{-i} (|\eta_1| + |\bar{\xi}_2| + \xi_1^2 + \bar{\eta}_2^2 + |\xi_1 \bar{\eta}_2|)); \end{aligned} \quad (48)$$

(Note that when deriving (48) we took into account that $\lambda^{2i} \leq \lambda^i \gamma^{-i}$, $\lambda_2^{2j} \leq \gamma^{-j} \lambda^j$.) According to Implicit Function Theorem we can resolve these equations with respect to variables $\bar{\xi}_2$ and η_1 . We obtain

$$\begin{aligned} \bar{\xi}_2 &= A\lambda^i x_1^+ (1 + \dots) + A\lambda^i (1 + \dots)\xi_1 + \lambda^j x_1^+ (1 + \dots)\bar{\eta}_2 + \\ &\quad \psi_1^{ij}(\xi_1, \eta_1), \\ \eta_1 &= A\lambda^j x_1^+ (1 + \dots) + A\lambda^j (1 + \dots)\bar{\eta}_2 + B\lambda^i x_1^+ (1 + \dots) \\ &\quad + B\lambda^i (1 + \dots)\xi_1 + \psi_2^{ij}(\xi_1, \eta_1); \end{aligned} \quad (49)$$

where

$$\psi_l^{ij}(\xi_1, \eta_1) = O(\lambda^i \gamma^{-i} (\xi_1^2 + \bar{\eta}_2^2 + |\xi_1 \bar{\eta}_2|)), \quad l = 1, 2.$$

It follows from (14) and (18) that the map $T_{21}T_{02}^j : (\bar{\xi}_2, \bar{\eta}_2) \mapsto (\bar{\bar{\xi}}_1, \bar{\bar{\eta}}_1)$ is written as

$$\begin{aligned} \bar{\bar{\xi}}_1 &= -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}\gamma^{-j}x_2^+(1 + \dots) + a_{21}\gamma^{-j}(1 + \dots)\bar{\xi}_2 + \\ &\quad (1 + \dots)\bar{\eta}_2 + \psi_3^{ij}(\bar{\xi}_2, \bar{\eta}_2), \\ \gamma^{-i}x_2^+(1 + \dots) + \gamma^{-i}\bar{\bar{\eta}}_1 \left(1 + \gamma^{-i}O(\bar{\bar{\xi}}_1 + \bar{\bar{\eta}}_1) \right) + \gamma^{-2i}O(\bar{\bar{\xi}}_1) &= \mu - \\ \gamma^{-j}x_2^+(1 + \dots) - \gamma^{-j}(1 + \dots)\bar{\xi}_2 + d_{21}(1 + \dots)\bar{\eta}_2^2 + \psi_4^{ij}(\bar{\xi}_2, \bar{\eta}_2); \end{aligned} \quad (50)$$

where the dots stand for the constant (i.e. independent on ξ and η) terms which tend to zero as $i, j \rightarrow \infty$. By ψ_3^{ij} and ψ_4^{ij} we denote the functions of the following orders:

$$\begin{aligned} \psi_3^{ij}(\bar{\xi}_2, \bar{\eta}_2) &= O(|\gamma^{-2j}|\bar{\xi}_2^2 + |\gamma^{-j}\bar{\xi}_2\bar{\eta}_2| + |\bar{\eta}_2^2|), \\ \psi_4^{ij}(\bar{\xi}_2, \bar{\eta}_2) &= O(|\gamma^{-2j}|\bar{\xi}_2^2 + |\gamma^{-j}\bar{\xi}_2\bar{\eta}_2| + |\bar{\eta}_2^3|). \end{aligned}$$

Let us substitute the expression for $\bar{\xi}_2$ and η_1 which is given by the system (49) into right side of the first equation of (50) and left side of the second equation of (50) respectively. Then, in new coordinates $\xi^{new} = \xi_1$ and $\eta^{new} = \bar{\eta}_2$ the first-return map $T_{21}T_{02}^j T_{12}T_{01}^i : (\xi, \eta) \mapsto (\bar{\bar{\xi}}, \bar{\bar{\eta}})$ can be written in following

form

$$\begin{aligned}
\bar{\bar{\xi}} &= -\frac{a_{21}}{2}\mu(1 + O(\mu)) + a_{21}\gamma^{-j}x_2^+(1 + \dots) + a_{21}A\lambda^i\gamma^{-j}(1 + \dots)\xi + \\
&\quad (1 + \dots)\eta + \psi_5^{ij}(\xi, \eta), \\
A\lambda^j\gamma^{-i}\bar{\bar{\eta}} \left(1 + \gamma^{-i}O(\bar{\bar{\xi}} + \bar{\bar{\eta}})\right) + \gamma^{-2i}O(\bar{\bar{\xi}}) &= \mu - \gamma^{-j}x_2^+(1 + \dots) + \\
A\lambda^i\gamma^{-j}(1 + \dots)\xi + O(\gamma^{-2i})\eta + d_{21}(1 + \dots)\eta^2 + \psi_6^{ij}(\xi, \eta);
\end{aligned} \tag{51}$$

where

$$\psi_5^{ij}(\xi, \eta) = O(|\lambda^i\gamma^{-i-j}|\xi^2 + |\lambda^i\gamma^{-j}\xi\eta| + |\eta^2|),$$

$$\psi_6^{ij}(\xi, \eta) = O(|\lambda^i\gamma^{-i-j}|\xi^2 + |\lambda^i\gamma^{-j}\xi\eta| + |\eta^3|).$$

(we took into account that $\lambda^i < \gamma^{-i}, \lambda^j < \gamma^{-j}, i < j, \dots$)

Now we substitute into the left-hand side of the second equation of (51) the expression for $\bar{\bar{\xi}}$ which is given by the first equation of (51). If we differentiate both sides of the obtained second equation with respect to η , then the resulting equation, since $d_{21} \neq 0$, defines η uniquely as a smooth function of ξ and $\bar{\bar{\eta}}$:

$$\eta = \varphi(\xi, \bar{\bar{\eta}}) = O(|\gamma^{-2i}| + |\gamma^{-j}|).$$

It is easy to see then, that if we write, schematically, the first equation of (51) in the form $\bar{\bar{\xi}} = \rho(\xi, \eta)$, then the system

$$\eta = \varphi(\xi, \eta), \quad \xi = \rho(\xi, \eta),$$

will have a unique solution (ξ^*, η^*) where

$$\xi^*, \eta^* = O(|\gamma^{-2i}| + |\gamma^{-j}| + |\mu|).$$

By construction, the following shift of the origin of the coordinates:

$$\xi^{new} = \xi - \xi^*, \quad \eta^{new} = \eta - \eta^*,$$

brings the system (51) to the form

$$\begin{aligned}
\bar{\bar{\xi}} &= a_{21}\gamma^{-j}x_2^+(1 + \dots) + a_{21}A\lambda^i\gamma^{-j}(1 + \dots)\xi + (1 + \dots)\eta + \psi_5^{ij}(\xi, \eta), \\
A\lambda^j\gamma^{-i}\bar{\bar{\eta}} \left(1 + \gamma^{-i}O(\bar{\bar{\xi}} + \bar{\bar{\eta}})\right) &= [\mu - \gamma^{-j}x_2^+ + \dots] + \\
A\lambda^i\gamma^{-j}(1 + \dots)\xi + d_{21}(1 + \dots)\eta^2 + \psi_6^{ij}(\xi, \eta);
\end{aligned} \tag{52}$$

Note that we have now nullified the constant (zero order) terms in (52) and in the first equation and the linear (first order) in η and $\bar{\bar{\xi}}$ term in the second equation of (52) is also made zero now. All zero order terms in the second equation of (52) are collected in the square brackets.

Now, we rescale variables as follows

$$\xi = \alpha X \quad \eta = \beta Y$$

where

$$\beta = -\gamma^{-i}\lambda^j\frac{A}{d_{21}}(1 + \dots) \quad \alpha = -\gamma^{-i}\lambda^j\frac{A}{d_{21}}(1 + \dots)$$

We obtain the following system:

$$\begin{aligned}\bar{X} &= Y + O(\lambda^i \gamma^{-j}), \\ \bar{Y}(1 + O(\lambda^i \gamma^{-j})) &= M(1 + \dots) - Y^2 + O((\lambda \gamma)^{i-j}).\end{aligned}\tag{53}$$

where

$$M = -[\mu - \gamma^{-i} x_2^+ + \dots] \frac{d_{21}}{A^2} \gamma^{2i} \lambda^{-2j}.$$

It is easy to verify that after small (order $O(\lambda^i \gamma^{-j})$) nonlinear transformations of variables U and V we can bring the system (53) to the form

$$\begin{aligned}\bar{X} &= Y \\ \bar{Y} &= M^{new} - Y^2 + O((\lambda \gamma)^{i-j}),\end{aligned}\tag{54}$$

where

$$M^{new} = -[\mu - \gamma^{-i} x_2^+ + \dots] \frac{d_{21}}{A^2} \gamma^{2i} \lambda^{-2j}.$$

The lemma is proven. ■

The map (45) is close to the one-dimensional parabola map

$$\bar{y} = M - y^2\tag{55}$$

whose bifurcations have been well studied. For the parabola map, the bifurcation set is contained in the interval $[-\frac{1}{4}, 2]$ of values of M : at $M = -\frac{1}{4}$ there exists a fixed point with multiplier equal to $+1$. This fixed point is attractive when $M \in (-\frac{1}{4}, \frac{3}{4})$ and it undergoes a period-doubling bifurcation at $M = \frac{3}{4}$.

Combining these results, with Lemma 7, we obtain that T_{ij} has similar bifurcations as the parabola map (55). The map has an attractive fixed point O_{ij} at μ from intervals $\delta(\mu)_i^{su}$ with boundary points

$$\begin{aligned}\mu_{ij}^{+1} &= x_2^+ \gamma^{-i} (1 + \dots) + \frac{A^2}{4d_{21}} \lambda^{2j} \gamma^{-2i} \\ \mu_{ij}^{-1} &= x_2^+ \gamma^{-i} (1 + \dots) - \frac{3A^2}{4d_{21}} \lambda^{2j} \gamma^{-2i}.\end{aligned}\tag{56}$$

which arises at the saddle-node bifurcation at $\mu = \mu_k^{+1}$ and loses stability at $\mu = \mu_k^{-1}$ through a period-doubling bifurcation. We thus obtain

Proposition 4 *In the parameter space there exist infinitely many intervals $\delta(\mu)_i^{su}$ accumulating to $\mu = 0$, such that for values of parameter μ from these intervals, the diffeomorphism f_μ has stable and unstable periodic points.*

4.4 Diffeomorphisms with infinitely many stable, unstable and elliptic periodic orbits in Newhouse regions

We now can combine the above results into the following proposition, providing also the proof of the fourth part of Theorem 1.

Proposition 5 *On δ_j^1 the values of μ , for which f_μ simultaneously has infinite many stable, and infinite many completely unstable and infinite many elliptic periodic orbits are dense.*

Proof Consider arbitrarily μ_j from interval δ_j . Then, it follows from lemma 4 that there exists, arbitrarily close to μ_j , a new $\mu = \mu_{j1}$ such that $f_{\mu_{j1}}$ has a non-transversal heteroclinic cycle $C_j = \{O_1, O_2, \Gamma_{12}, \Gamma_{21j}\}$. It follows from proposition 3 that in any neighbourhood of the point μ_{j1} there exists an interval $\delta_{1,0}^{(j)} \in \delta_j$ of values of μ such that for $\mu \in \delta_{1,0}^{(j)}$ the family f_μ has an elliptic periodic orbit.

Furthermore, on the interval $\delta_{1,0}^{(j)}$, by virtue of proposition 4 and density of diffeomorphisms with non-transversal heteroclinic cycle, there exists an interval $\delta_{1,1}^{(j)} \subset \delta_{1,0}^{(j)}$ such that for $\mu \in \delta_{1,1}^{(j)}$ the family f_μ has elliptic and stable—unstable periodic orbits, which lie in U . Similarly, on the interval $\delta_{1,1}^{(j)}$ we find a subinterval $\delta_{2,2}^{(j)}$ such that for $\mu \in \delta_{2,2}^{(j)}$ the diffeomorphism f_μ has two elliptic orbits, two completely unstable and two stable periodic orbits lying in U .

Finally we obtain a countable set of nested intervals

$$\delta_{1,1}^{(j)} \supset \delta_{2,2}^{(j)} \supset \dots \supset \delta_{n,n}^{(j)} \supset \dots$$

such that for $\mu \in \delta_{n,n}^{(j)}$ the diffeomorphism f_μ simultaneously has n elliptic, n stable and n completely unstable periodic orbits. This completes the proof of Proposition 5, and in turn of Theorem 1. ■

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