Geometric properties for the design of unusual member cross-sections in bending

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Abstract

This short communication presents a series of general geometric properties of unusually-shaped member cross-sections. A skeleton derivation of the most important properties is presented and a method is illustrated for constructing more complex cross-sections. The focus is on the cross-sectional area, centroid and second moment of area, used in classical engineering design for bending. The shapes presented include circular and elliptical segments, sectors and arcs, semiellipses, superellipses, and combinations thereof. The superellipse allows a vast range of novel cross-sections to be described analytically and is a powerful mathematical tool for the analysis of unusual curved shapes. The equations are presented as composites of their geometric components so as to facilitate practicing engineers in their implementation as programmed functions or subroutines.

Keywords

Bending design; complex cross-sections; lightweight structures; second moment of area; elliptical geometry; superellipse.

1. Introduction

A series of section properties for non-standard, mono-symmetric, thick-walled, curved hollow beam cross-sections are presented in what follows. The target audience of this work is practicing engineers in the light-weight construction, aerospace and sports industries undertaking design and optimisation of unusually-shaped members in bending, usually from cold-formed steel or other ductile material (Boljanovic, 2009). The shapes presented are circular and elliptical segments, sectors and arcs, semiellipses and superellipses, which when combined allow a very wide range of curved cross-sectional geometries to be analysed. The section properties presented are those required to obtain the radii of gyration and elastic section moduli, used in classical engineering design for bending, notably the cross-sectional area, section centroid and second moment of area (Ghali *et al.*, 2008).

An outline derivation of standard results is presented first, followed by an illustration of how these may be subsequently used as building blocks to construct increasingly complex cross-sectional geometries. Extensive use is made of the parallel axis theorem (Ghali *et al.*, 2008). The final equations are presented as composite functions of their geometric components, as they would otherwise be very protracted algebraically, in such a way as to facilitate their implementation as programmed functions or subroutines.

2. Standard results for reference shapes

2.1 Circular and elliptical segments

Figure 1 shows a generalised elliptical segment with major and minor axes a and b respectively, angle $2\theta_G$ such that $\theta_G \leq \frac{1}{2}\pi$ and centroid located at $(x, y) = (0, y_C)$. Consider an infinitesimally small element of area dA , oriented either vertically or horizontally. The area, centroid and second moment of area about the x and y axes of the full segment may be calculated by $A = \int dA$, $y_C = \int yx dy / \int dA$, $I_x = \int y^2 dA = \int y'^2 x dy$ and $I_y = \int x^2 y' dx$ respectively, where $x = a \sin \theta$, $y = b \cos \theta$ and $y' = b(\cos \theta_G - \cos \theta)$. The angle θ is defined from the vertical y axis in the range $-\theta_G \leq \theta \leq \theta_G$. Integrating over the appropriate range of θ yields the following general properties:

$$A_{seg}(a, b, \theta_G) = \frac{1}{2} ab (2\theta_G - \sin 2\theta_G) \quad (1)$$

$$y_{C,seg}(b, \theta_G) = \frac{4b \sin^3 \theta_G}{3(2\theta_G - \sin 2\theta_G)} \quad (2)$$

$$I_{x,seg}(a, b, \theta_G) = \frac{ab^3}{16}(4\theta_G - \sin 4\theta_G) \quad (3)$$

$$I_{y,seg}(a, b, \theta_G) = \frac{a^3b}{24}(6\theta_G - \sin 2\theta_G(3 - 2\sin^2 \theta_G)) \quad (4)$$

These equations simplify to those for a circular segment when $a = b = \rho$, where ρ is the radius. The second moment of area about the horizontal centroidal C axis I_C is obtained by application of the parallel axis theorem: $I_C = I_x - Ay_C^2$. The distances from the centroid to the extreme fibres of the beam are given by x_1, x_2, y_1 and y_2 and used to calculate the elastic section moduli: $Z_{x1,2} = I_C / y_{1,2}$ and $Z_{y1,2} = I_y / x_{1,2}$. The radii of gyration are thus given by $r_x = (I_C / A)^{1/2}$ and $r_y = (I_y / A)^{1/2}$.

Fig. 1 – General elliptical segment geometry

2.2 Circular and elliptical sectors

The section properties of an elliptical sector with major and minor axes a and b respectively and angle $2\theta_G$ such that $\theta_G \leq \frac{1}{2}\pi$ (Fig. 2) are easily constructed from those of a segment (Fig. 1) and an isosceles triangle with dimensions $c = 2a\sin\theta_G$ and $d = b\cos\theta_G$. The triangle thus has area $A = \frac{1}{2}cd = ab\sin\theta_G\cos\theta_G$, centroid $y_{C,tr} = \frac{2}{3}d = \frac{2}{3}b\cos\theta_G$ and second moment of area about the x and y axes $I_x = \frac{1}{4}cd^3 = \frac{1}{4}ab^3\sin\theta_G\cos^3\theta_G$ and $I_y = \frac{1}{48}c^3d = \frac{1}{6}a^3b\sin^3\theta_G\cos\theta_G$ respectively. The properties of the elliptical sector are thus:

$$A_{sec}(a, b, \theta_G) = ab\theta_G \quad (5)$$

$$y_{C,sec}(b, \theta_G) = \frac{2b \sin \theta_G}{3\theta_G} \quad (6)$$

$$I_{x,sec}(a, b, \theta_G) = \frac{ab^3}{8} (2\theta_G + \sin 2\theta_G) \quad (7)$$

$$I_{y,sec}(a, b, \theta_G) = \frac{a^3b}{8} (2\theta_G - \sin 2\theta_G) \quad (8)$$

These equations simplify to those for a circular sector when $a = b = \rho$, where ρ is the radius.

Fig. 2 – General elliptical sector geometry

2.3 The superellipse

The elliptical segment (Fig. 1) may be generalised into the parametric form $x = a \sin^{(2/p)} \theta$ and $y = b \cos^{(2/q)} \theta$, such that p and $q > 0$, known as the superellipse (Gridgeman, 1970). This is a powerful generalisation which allows great freedom to analyse a vast range of curved cross-sectional geometries. Some example shapes are illustrated in Fig. 3. A closed-form analytical solution is only possible for a semisuperellipse ($\theta_G = \frac{1}{2}\pi$):

$$A_{sup}(a, b, p, q) = \frac{2ab}{q} \frac{\Gamma\left(\frac{1}{q}\right) \Gamma\left(\frac{1+p}{p}\right)}{\Gamma\left(\frac{p+pq+q}{pq}\right)} \quad (9)$$

$$y_{C,sup}(b, p, q) = \frac{4^{\frac{1}{q}} b}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{2+q}{2q}\right) \Gamma\left(\frac{p+pq+q}{pq}\right)}{\Gamma\left(\frac{2p+pq+q}{pq}\right)} \quad (10)$$

$$I_{x,sup}(a, b, p, q) = \frac{2ab^3}{q} \frac{\Gamma\left(\frac{3}{q}\right) \Gamma\left(\frac{1+p}{p}\right)}{\Gamma\left(\frac{3p+pq+q}{pq}\right)} \quad (11)$$

$$I_{y,sup}(a, b, p, q) = \frac{2a^3b}{p} \frac{\Gamma\left(\frac{3}{p}\right) \Gamma\left(\frac{1+q}{q}\right)}{\Gamma\left(\frac{p+pq+3q}{pq}\right)} \quad (12)$$

where $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$ is the gamma function.

The gamma function (Temme, 1996) must be evaluated numerically but it is easily invoked in common software packages, including Excel, Matlab, Maple and Mathematica. Note that if $p = q = 2$, Eqs 9 to 12 reduce to Eqs 1 to 4 respectively evaluated at $\theta_G = \frac{1}{2}\pi$. Further, in the limit p or $q \rightarrow \infty$, the section properties become those of a perfect rectangle:

$$\lim_{p \rightarrow \infty} A_{sup} = \lim_{q \rightarrow \infty} A_{sup} = 2ab \text{ and } \lim_{p \rightarrow \infty} y_{C,sup} = \lim_{q \rightarrow \infty} y_{C,sup} = \frac{1}{2}b$$

$$\lim_{p \rightarrow \infty} I_{x,sup} = \lim_{q \rightarrow \infty} I_{x,sup} = \frac{2}{3}ab^3 \text{ and } \lim_{p \rightarrow \infty} I_{y,sup} = \lim_{q \rightarrow \infty} I_{y,sup} = \frac{2}{3}a^3b$$

The same general mathematical framework may thus be employed to analyse beam sections with any combinations of curved and rectangular geometric components.

Fig. 3 – Superellipses as a function of the powers p and q . Left: typical shapes. Right: more unusual shapes.

3. Properties of simple open-section beams

3.1 Simple elliptical arc cross-section

The section properties of a simple cross-section composed of an elliptical arc with finite wall thickness t (Fig. 4) are obtained by subtracting two elliptical segments. The outer and inner segments have major and minor axes a and b , and $a' = a - t$ and $b' = b - t$, respectively. The angles to the vertical θ_G and θ'_G are related by $\theta'_G = \cos^{-1}[a \cos \theta_G / (a - t)]$, so that the arc has a horizontal edge. For a circular arc, $a = b = \rho$, where ρ is the radius. The section properties are as follows:

$$A = A_{seg}(a, b, \theta_G) - A_{seg}(a', b', \theta'_G) \quad (13)$$

$$y_C = \frac{1}{A} [y_{C,seg}(b, \theta_G) A_{seg}(a, b, \theta_G) - y_{C,seg}(b', \theta'_G) A_{seg}(a', b', \theta'_G)] \quad (14)$$

$$I_C = I_{x,seg}(a, b, \theta_G) - I_{x,seg}(a', b', \theta'_G) - A y_C^2 \quad (15)$$

$$I_y = I_{y,seg}(a, b, \theta_G) - I_{y,seg}(a', b', \theta'_G) \quad (16)$$

Extreme fibre distances: $y_1 = b - y_C$, $y_2 = y_C - b \cos \theta_G$ and $x_{1,2} = a \sin \theta_G$.

Fig. 4 – Elliptical arc cross-section (variant 1 – horizontal edge)

A second and perhaps more common version of this cross-section assumes that the arc edge is inclined at an angle θ_G to the vertical (Fig. 5). This section is obtained by subtracting two elliptical sectors. The beam properties are thus similarly calculated according to Eqs 13 to 16, though using A_{sec} , $y_{C,sec}$, $I_{x,sec}$ and $I_{y,sec}$ (Eqs 5 to 8) instead of A_{seg} , $y_{C,seg}$, $I_{x,seg}$ and $I_{y,seg}$ respectively. For an alternative thin-walled solution (assuming $t \ll a$ and $t \ll b$), the section properties of this common cross-section may be closely approximated as follows:

$$A \approx (a+b)t\theta_G \quad (17)$$

$$y_C \approx \frac{(2ab+b^2-t(a+2b))}{a+b-t} \frac{2\sin\theta_G}{3\theta_G} \quad (18)$$

$$I_C \approx \frac{1}{8}t(3ab^2+b^3-3tb(a+b))(2\theta_G+\sin 2\theta_G)-\dots \\ \frac{tb(b+2a)(b^2+2ab-2t(a+2b))}{a+b-t} \frac{4\sin^2\theta_G}{9\theta_G} \quad (19)$$

$$I_y \approx \frac{1}{8}t(a^3+3a^2b-3ta(a+b))(2\theta_G-\sin 2\theta_G) \quad (20)$$

The case of a circular arc, $a = b = \rho$, simplifies each of the above considerably. The extreme fibre distances are: $y_1 = b - y_C$, $y_2 = y_C - (b - t)\cos\theta_G$ and $x_{1,2} = a\sin\theta_G$.

Fig. 5 – Elliptical arc cross-section (variant 2 – inclined edge)

3.2 Semielliptical cross-section

A semielliptical cross-section of finite wall thickness t (Fig. 6) is obtained by subtracting two semiellipses ($\theta_G = \frac{1}{2}\pi$) with major axes a and $a' = a - t$ and minor axes b and $b' = b - t$ respectively. Further, the shape of the cross-section may be closely controlled by the parameters p and q to generate hollow curved cross-sections of virtually any desired shape (i.e. Fig. 3). Setting $p = q = 2$ generates a normal semiellipse, and additionally $a = b = \rho$ generates a semicircle. The section properties are as follows:

$$A = A_{sup}(a, b, p, q) - A_{sup}(a', b', p, q) \quad (21)$$

$$y_C = \frac{1}{A} [y_{C,sup}(b, p, q) A_{sup}(a, b, p, q) - y_{C,sup}(b', p, q) A_{sup}(a', b', p, q)] \quad (22)$$

$$I_C = I_{x,sup}(a, b, p, q) - I_{x,sup}(a', b', p, q) - A y_C^2 \quad (23)$$

$$I_y = I_{y,sup}(a, b, p, q) - I_{y,sup}(a', b', p, q) \quad (24)$$

Extreme fibre distances: $y_1 = b - y_C$, $y_2 = y_C$ and $x_{1,2} = a$.

Fig. 6 – Semielliptical cross-section

4. Example of a more complex open cross-section

This section presents an example of how a more complex curved cross-section may be constructed using the general shapes presented earlier as building blocks. The cross-section shown in Fig. 7, for example, is a variant of the circular arc where the tip of the arc has been flattened up to a deformation δ . The flattening itself may be assumed to be superelliptical in cross-section and thus may be controlled by adjusting the parameters p and q (Fig. 3).

Fig. 7 – Cross-section of a deformed circular arc cross-section with superelliptical flattening at the tip

The section properties are derived in this case by subtracting a circular arc with angle $2\theta_S$ from a circular arc with angle $2\theta_G$ (so that $\theta_S < \theta_G$), both at origin O , and putting a semiellipse with origin at O' in its place. The circular portion of the arc is obtained from two circular segments with origin at O , assuming radii ρ and $\rho' = \rho - t$ for the outer and inner arcs respectively. The outer and inner semiellipses have major and minor axes $a = \rho \sin \theta_S$ and $a' = (\rho - t) \sin \theta'_S$, and $b = \rho(1 - \cos \theta_S) - \delta$ and $b' = b - t$, respectively. The following relations are also required:

$$\theta'_G = \cos^{-1} \left[\frac{\rho}{\rho - t} \cos \theta_G \right] \text{ and } \theta'_S = \cos^{-1} \left[\frac{\rho}{\rho - t} \cos \theta_S \right]$$

The section properties are thus as follows:

$$A_{out,circ} = A_{seg}(\rho, \rho, \theta_G) - A_{seg}(\rho, \rho, \theta_S)$$

$$A_{in,circ} = A_{seg}(\rho', \rho', \theta'_G) - A_{seg}(\rho', \rho', \theta'_S)$$

$$A_{out,ell} = A_{sup}(a, b, p, q)$$

$$A_{in,ell} = A_{sup}(a', b', p, q)$$

$$A = [A_{out,circ} + A_{out,ell}] - [A_{in,circ} + A_{in,ell}] \quad (25)$$

$$y_{C,out,circ} = \frac{1}{A_{out,circ}} [y_{C,seg}(\rho, \theta_G) A_{seg}(\rho, \rho, \theta_G) - y_{C,seg}(\rho, \theta_S) A_{seg}(\rho, \rho, \theta_S)]$$

$$y_{C,in,circ} = \frac{1}{A_{in,circ}} [y_{C,seg}(\rho', \theta'_G) A_{seg}(\rho', \rho', \theta'_G) - y_{C,seg}(\rho', \theta'_S) A_{seg}(\rho', \rho', \theta'_S)]$$

$$y_{C,out,ell} = y_{C,seg}(b, p, q) + \rho \cos \theta_S$$

$$y_{C,in,ell} = y_{C,seg}(b', p, q) + \rho' \cos \theta'_S$$

$$y_C = \frac{1}{A} \left[\left(y_{C,out,circ} A_{out,circ} + y_{C,out,ell} A_{out,ell} \right) - \dots \right] \quad (26)$$

$$I_{C,circ} = [I_{x,seg}(\rho, \rho, \theta_G) - I_{x,seg}(\rho, \rho, \theta_S)] - [I_{x,seg}(\rho', \rho', \theta'_G) - I_{x,seg}(\rho', \rho', \theta'_S)] - \dots$$

$$(A_{out,circ} - A_{in,circ}) y_{C,circ}^2$$

$$I_{C,ell} = I_{x,sup}(a, b, p, q) - I_{x,sup}(a', b', p, q) - (A_{out,ell} - A_{in,ell}) (y_{C,ell} - \rho \cos \theta_S)^2$$

$$y_{C,circ} = \frac{1}{(A_{out,circ} - A_{in,circ})} [y_{C,out,circ} A_{out,circ} - y_{C,in,circ} A_{in,circ}]$$

$$y_{C,ell} = \frac{1}{(A_{out,ell} - A_{in,ell})} [y_{C,out,ell} A_{out,ell} - y_{C,in,ell} A_{in,ell}]$$

$$I_C = \left[I_{C,circ} + (A_{out,circ} - A_{in,circ}) (y_C - y_{C,circ})^2 \right] + \dots \quad (27)$$

$$\left[I_{C,ell} + (A_{out,ell} - A_{in,ell}) (y_C - y_{C,ell})^2 \right]$$

$$I_{y,out} = I_{y,seg}(\rho, \rho, \theta_G) - I_{y,seg}(\rho, \rho, \theta_S) - I_{y,sup}(a, b, p, q)$$

$$I_{y,in} = I_{y,seg}(\rho', \rho', \theta'_G) - I_{y,seg}(\rho', \rho', \theta'_S) - I_{y,sup}(a', b', p, q)$$

$$I_y = I_{y,out} - I_{y,in} \quad (28)$$

Extreme fibre distances: $y_1 = \rho - \delta - y_C$, $y_2 = y_C - \rho \cos \theta_G$ and $x_{1,2} = \rho \sin \theta_G$.

5. Extension of geometric properties to a closed cross-section

Open cross-sections which are extended to half of the circumference ($\theta_G = \frac{1}{2}\pi$) may be combined together to make a hollow closed cross-section with finite thickness t , e.g. a tube, hollow rod, pipe or cylindrical shell, as illustrated in Fig. 8. The x axis is assumed to be at the junction between any two shapes. Combining superelliptical cross-sections in this way substantially expands the range of beam sections which may now be solved analytically. Assigning subscripts 1 and 2 to the relevant properties of the upper or lower part respectively, the section properties of the full closed cross-section are as follows:

$$A = A_1 + A_2 \quad (29)$$

$$y_C = \frac{1}{A} [A_1 (y_{C1} + b_2) + A_2 (b_2 - y_{C2})] \quad (30)$$

$$I_C = [I_{C1} + A_1 (y_{C1} + b_2 - y_C)^2] + [I_{C2} + A_2 (y_C - b_2 + y_{C2})^2] \quad (31)$$

$$I_y = I_{y1} + I_{y2} \quad (32)$$

Extreme fibre distances: $y_1 = d - y_C$, $y_2 = y_C$ and $x_{1,2} = a$.

Fig. 8 – Extension of section properties to a closed-section beam

6. Conclusions

This short paper has presented a series of highly general geometric properties of unusual beam cross-sections in the context of classical engineering design for bending. The equations presented here go substantially beyond the standard results that are typically available to practicing engineers, and may be applied to both open and closed finite thickness hollow cross-sections. Furthermore, the application of the superelliptical shape in engineering design is a powerful concept as it offers a closed-

form analytical solution which can be employed readily to analyse and optimise a wide range of complex geometries. A similar methodology may be used to derive corresponding general properties for torsion and shear.

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