1 **Approximate solutions for Forchheimer** 2 **flow to a well**

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4 November 20, 2007

5 **Abstract**

6 An exact solution for transient Forchheimer flow to a well does not cur-
7 rently exist. However, this paper presents a set of approximate solutions,
8 which can be used as a framework for verifying future numerical models
9 that incorporate Forchheimer flow to wells. These include: a large time ap-
10 proximation derived using the method of matched asymptomatic expansion,
11 a Laplace transform approximation of the well-bore response, designed to
12 work well when there is significant well-bore storage and flow is very turbu-
13 lent; and a simple heuristic function for when flow is very turbulent and the
14 well radius can be assumed infinitesimally small. All the approximations
15 are compared to equivalent finite difference solutions.

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16 Key-words: finite difference, Forchheimer equation, Laplace transform, non-
17 Darcy flow, similarity transform, well hydraulics.

18 **1 Introduction**

19 Non-Darcian post-linear flow has been observed in numerous hydraulic experi-
20 ments in both coarse granular media (Thiruvengadam and Pradip Kumar, 1997;
21 Venkataraman and Rama Mohan Rao, 1998, 2000; Legrand, 1999; Chen et al.,
22 2003; Reddy and Rama Mohan Rao, 2006; Sidiropoulou et al., 2007) and frac-
23 tured media (Kohl et al., 1997; Lee and Lee, 1999; Qian et al., 2005, 2007). Non-
24 Darcian flow is often distinguished as being either pre- or post-linear flow. Pre-
25 linear flow typically occurs at low Reynolds' numbers (Firdaouss et al., 1997).
26 In this paper, we are concerned with post-linear flow, which conversely occurs at
27 high Reynold's numbers (Zeng and Grigg, 2006). This is of particular concern in
28 close proximity to abstraction wells where flow velocities are enhanced due to the
29 convergence of flow lines (Sen, 1988, 1990; Kohl et al., 1997; Ewing et al., 1999;
30 Ewing and Lin, 2001; Kelkar, 2000; Kolditz, 2001; Wu, 2002a,b).

31 A popular method for representing the post-linear regime is to exchange Darcy's
32 Law with Forchheimer's equation (Forchheimer, 1901). There is both a theoretical
33 (Irmay, 1958; Whitaker, 1996; Giorgi, 1997; Chen et al., 2001) and experimental
34 (Thiruvengadam and Pradip Kumar, 1997; Kohl et al., 1997; Venkataraman and
35 Rama Mohan Rao, 1998, 2000; Reddy and Rama Mohan Rao, 2006; Sidiropoulou
36 et al., 2007) basis for doing this.

37 Bear (1979, p.308) and Ewing et al. (1999) obtain an exact solution for steady
38 state radial Forchheimer flow to a well. Kelkar (2000) and Wu (2002a) present

39 an approximate solution for transient radial Forchheimer flow to a well, which is
40 suitable for large times. Moutsopoulos and Tsihrintzis (2005) obtain an approx-
41 imate similarity solution for one-dimensional transient Forchheimer flow, which
42 works well for large flow rates. Wen et al. (2006) obtain an exact similarity so-
43 lution for one-dimensional non-Darcian flow using the Izbash (1931) equation,
44 which assumes that water flux is related to hydraulic head by a power law. Wen
45 et al. (2007) present an approximate Laplace transform solution for transient radial
46 Izbash flow, which works well at large times.

47 Meanwhile, Sen (1988) claims to have derived a similarity solution for tran-
48 sient radial Forchheimer flow to an infinitesimal well using the Boltzmann trans-
49 form. This solution has been extended by Sen (1989) to consider Izbash flow,
50 by Sen (1990) to consider large-diameter wells and by Birpinar and Sen (2004)
51 to consider leaky aquifers. Indeed, for Darcian flow to an infinitesimal well, hy-
52 draulic head is a function of the Boltzmann transform (e.g. Theis, 1935). How-
53 ever, Camacho-V. and Vasquez-C. (1992) explain that this is not the case for non-
54 Darcian flow and therefore the solution is invalid. Sen (1992) dismisses this claim
55 on the basis of insufficient evidence.

56 Exact and approximate mathematical solutions such as those discussed above
57 generally represent highly idealized situations. In order to look at more realistic
58 cases, such as when there is significant drainage from an unsaturated zone (Dogan
59 and Motz, 2005) or a seepage face (Rushton, 2006) or when the abstraction well
60 geometry is particularly complex (Demir and Narasimhan, 1994), it is often nec-
61 essary to use a numerical model (Narasimhan, 2007). Nevertheless, mathematical
62 solutions are invaluable for model verification.

63 The idea behind verifying a numerical model with a mathematical solution is

64 that the two should produce identical results for the prescribed scenario (e.g. Zop-
65 pou and Roberts, 2003; Simpson and Clement, 2004; Dogan and Motz, 2005).
66 Whereas there is a vast wealth of appropriate solutions for Darcian flow to well
67 problems (e.g. Moench, 1997; Mathias and Butler, 2006, 2007a,b), for Forch-
68 heimer flow, with the exception of the steady state solution of Bear (1979, p.308)
69 and Ewing et al. (1999), there are only the approximate solutions of Sen (1990),
70 Kelkar (2000) and Wu (2002a). Unfortunately, the derivations of these two solu-
71 tions are non-rigorous and therefore cannot be guaranteed to properly reconcile
72 with a correctly functioning numerical model.

73 The outline of this paper is as follows: a finite difference solution for transient
74 radial Forchheimer flow to a well is developed; the derivation of Sen's similarity
75 solution is examined in detail; a rigorous derivation for the approximate solution
76 of Kelkar (2000) and Wu (2002a) is obtained using the method of matched asymp-
77 totic expansion; the method of Wen et al. (2007) is used to derive an approximate
78 Laplace transform solution for Forchheimer flow to a well, which is valid for large
79 times and large flow rates; a heuristic function is then proposed for Forchheimer
80 flow to an infinitesimal well and is shown to work well for all times providing
81 the flow rate is very large. The limitations of the four approximate mathematical
82 solutions are explored by comparison with a numerical solution obtained using fi-
83 nite differences. The purpose of this paper is to provide a framework for verifying
84 future numerical models that incorporate Forchheimer flow to wells.

85 **2 The governing equations**

86 The governing equation of flow to a fully penetrating well in a homogenous,
87 isotropic and confined aquifer is (Papadopoulos and Cooper, 1967)

$$S_s \frac{\partial \phi}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (rq) = 0 \quad (1)$$

88 subjected to the initial and boundary conditions:

$$\begin{aligned} \phi &= 0, & r &\geq r_w, & t &= 0 \\ \phi &= \phi_w, & r &= r_w, & t &> 0 \\ \phi &= 0, & r &= \infty, & t &> 0 \end{aligned} \quad (2)$$

89 where S_s [L^{-1}] is the specific storage coefficient, ϕ [L] is hydraulic head, t [T]
90 is time, r [L] is the radial distance from the well, ϕ_w [L] is the hydraulic head in
91 the well-bore, r_w [L] is the well radius and q [LT^{-1}] is the water flux, assumed
92 here to be found from Forchheimer's equation (Forchheimer, 1901)

$$q + \beta q^2 = -K \frac{\partial \phi}{\partial r} \quad (3)$$

93 where K [LT^{-1}] is the hydraulic conductivity and β [$L^{-1}T$] is the turbulent
94 flow coefficient.

95 The equation for the well-bore is (Papadopoulos and Cooper, 1967)

$$\pi r_c^2 \frac{d\phi_w}{dt} + Q + 2\pi m r_w q(r = r_w) = 0 \quad (4)$$

96 subjected to

$$\phi_w = 0, \quad t = 0 \quad (5)$$

where Q [L^3T^{-1}] is the pumping rate from the well, which is positive for abstraction, r_c [L] is the radius of the well casing and m [L] is the aquifer thickness.

3 Dimensionless transformation

Applying the following dimensionless transformations:

$$\phi_D = -\frac{2\pi m K \phi}{Q}, \quad \phi_{wD} = -\frac{2\pi m K \phi_w}{Q}, \quad t_D = \frac{Kt}{S_s m^2} \quad (6)$$

$$q_D = -\frac{2\pi m^2 q}{Q}, \quad \beta_D = -\frac{Q\beta}{2\pi m^2} \quad (7)$$

$$r_D = \frac{r}{m}, \quad r_{wD} = \frac{r_w}{m}, \quad r_{cD} = \frac{r_c}{S_s^{1/2} m^{3/2}} \quad (8)$$

the above problem reduces to:

$$\frac{\partial \phi_D}{\partial t_D} + \frac{1}{r_D} \frac{\partial}{\partial r_D} (r_D q_D) = 0 \quad (9)$$

$$q_D + \beta_D q_D^2 = -\frac{\partial \phi_D}{\partial r_D} \quad (10)$$

$$\begin{aligned}
\phi_D &= 0, & r_D &\geq r_{wD}, & t_D &= 0 \\
\phi_D &= \phi_{wD}, & r_D &= r_{wD}, & t_D &> 0 \\
\phi_D &= 0, & r_D &= \infty, & t_D &> 0
\end{aligned} \tag{11}$$

$$\frac{r_{cD}^2}{2} \frac{d\phi_{wD}}{dt_D} - 1 + r_{wD} q_D(r_D = r_{wD}) = 0 \tag{12}$$

$$\phi_{wD} = 0, \quad t_D = 0 \tag{13}$$

102 4 Finite difference solution

103 Numerical models have been developed for Forchheimer flow to a well using both
104 finite differences (Choi et al., 1997; Ewing and Lin, 2001; Wu, 2002a,b) and finite
105 elements (Ewing et al., 1999; Ewing and Lin, 2001; Kolditz, 2001). In this paper
106 we use finite differences. We start by discretizing the radial axis r_D into N number
107 of nodes such that $r_{wD} < r_i < r_{eD}$ for $i = 1 \dots N$ where, r_i is the value of r_D at the
108 i th node and r_{eD} is a large radial distance from the well at which to approximate
109 the boundary condition at $r_D = \infty$. The dimensionless head, ϕ_D is approximated
110 at each node by ϕ_i . Having discretized in space, the above problem reduces to the
111 following set of ordinary differential equations with respect to time:

$$\frac{d\phi_i}{dt_D} \approx \frac{r_{i-1/2} q_{i-1/2} - r_{i+1/2} q_{i+1/2}}{r_i(r_{i+1/2} - r_{i-1/2})}, \quad i = 1 \dots N \tag{14}$$

112 where (Wu, 2002b)

$$q_{i-1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{\phi_{i-1} - \phi_i}{r_i - r_{i-1}} \right) \right]^{1/2} \right\}, \quad i = 2 \dots N \quad (15)$$

$$q_{i+1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{\phi_i - \phi_{i+1}}{r_{i+1} - r_i} \right) \right]^{1/2} \right\}, \quad i = 1 \dots N-1 \quad (16)$$

113 The boundary conditions are implemented through:

$$q_{1-1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{\phi_{wD} - \phi_1}{r_1 - r_{wD}} \right) \right]^{1/2} \right\} \quad (17)$$

$$q_{N+1/2} = \frac{1}{2\beta_D} \left\{ -1 + \left[1 + 4\beta_D \left(\frac{\phi_N - 0}{r_{eD} - r_N} \right) \right]^{1/2} \right\} \quad (18)$$

114 where ϕ_{wD} , is the dimensionless head in the well-bore, which is approximated
115 by

$$\frac{d\phi_{wD}}{dt_D} \approx \frac{2}{r_{cD}^2} [1 - r_{wD} q_{1-1/2}] \quad (19)$$

116 The above set of equations are integrated with respect to time using the stiff
117 integrator ODE15s (Shampine and Reichelt, 1997; Shampine et al., 1999) avail-
118 able in any standard version of MATLAB. Due to the convergence of flow lines
119 at the well, it is a good idea to space the nodes logarithmically in the r_D direction
120 (Wu, 2002b) such that

$$r_i = (r_{i-1/2} + r_{i+1/2})/2, \quad i = 1 \dots N \quad (20)$$

121 where

$$\log_{10}(r_{i+1/2}) = \log_{10}(r_w) + i \left[\frac{\log_{10}(r_{eD}) - \log_{10}(r_{wD})}{N} \right], i = 0 \dots N \quad (21)$$

For all the simulations presented in this paper, r_{wD} and r_{cD} were both set to 1. The observation well response was found to be insensitive to the abstraction well diameter, r_{wD} when $r_D \geq 10^3$, therefore, when presenting results for an infinitesimal diameter abstraction well, the location of the observation well was set to a normalized distance of $r_D = 10^3$. Both the observation and abstraction well responses were then found to be insensitive to the far-field boundary condition when r_{eD} was set to 10^8 . From a grid sensitivity study, it was found sufficient to set $N = 2000$ nodes. A specified time-step is not needed as ODE15s uses an adaptive time grid.

5 Sen's solution

Sen (1988) attempts to obtain a similarity solution by substituting the independent variable transform (IVT), $\xi = r_D^2/t_D$ (i.e. Boltzmann transform) into equations (9) and (10) to obtain the ordinary differential equation

$$\frac{dq_D}{d\xi} + \left(\frac{1}{4} + \frac{1}{2\xi} \right) q_D + \frac{\beta_D}{4} q_D^2 = 0 \quad (22)$$

which has the general solution (Sen, 1988)

$$q_D = \frac{e^{-\xi/4}}{\xi^{1/2}} \left[A + \beta_D \frac{\pi^{1/2}}{2} \operatorname{erf} \left(\frac{\xi^{1/2}}{2} \right) \right]^{-1} \quad (23)$$

136 where A is an integration constant, dependant on the boundary condition at
 137 $\xi = 0$ and the erf operator denotes the error function.
 138 Sen (1988) considers an infinitesimal well in an infinite aquifer and therefore
 139 applies the initial and boundary conditions:

$$\begin{aligned} \phi_D &= 0, & r_D &\geq 0, & t_D &= 0 \\ r_D q_D &= 1, & r_D &= 0, & t_D &> 0 \\ \phi_D &= 0, & r_D &= \infty, & t_D &> 0 \end{aligned} \quad (24)$$

140 The transformed boundary condition is

$$\lim_{\xi \rightarrow 0} \xi^{1/2} q_D = t_D^{-1/2} \quad (25)$$

141 A crucial requirement for the applicability of similarity arguments is that both
 142 the governing equations and all the initial and boundary conditions be reducible
 143 to similarity form (e.g. Kevorkian, 1990, p.8). The presence of the $t_D^{-1/2}$ term
 144 in equation (25) shows that this is not the case, which supports the concern of
 145 Camacho-V. and Vasquez-C. (1992) that Sen's solution is not valid. Nevertheless,
 146 from equation (25), Sen (1988) concludes that $A = t_D^{1/2}$ and therefore that

$$q_D = \frac{e^{-\xi/4}}{\xi^{1/2}} \left[t_D^{1/2} + \beta_D \frac{\pi^{1/2}}{2} \operatorname{erf} \left(\frac{\xi^{1/2}}{2} \right) \right]^{-1} \quad (26)$$

147 Finally, Sen (1988) obtains an expression for ϕ_D by rearranging equation (10)
 148 such that

$$\phi_D = \int_{r_D}^{\infty} (q_D + \beta_D q_D^2) dr_D \quad (27)$$

Figure 1 compares the Sen (1988) solution given in equation (27) with the finite difference solution developed in section 4. The integral in equation (27) was evaluated using an adaptive Lobatto quadrature (the `quadl` command in MATLAB). Despite Sen's non-rigorous handling of the independent variable transform, the solution approximates the finite difference solution relatively well. Nevertheless, for small times it overestimates and for intermediate times it underestimates. However, because both the solutions are essentially approximate it is not yet possible to say which one is more accurate.

6 Solution by matched asymptotic expansions

A popular method for solving non-linear partial differential equations is the method of matched asymptotic expansions (e.g. Kevorkian, 1990, p.478). At large times, the head profile has spread out over a large distance. Roose et al. (2001) were interested in a similar mathematical scenario but in the context of nutrient uptake in cylindrical plant roots. Following Roose et al. (2001), this can be specified by writing

$$t_D = \frac{\tau}{\beta_D^2} \quad \text{and} \quad r_D = \frac{R}{\beta_D} \quad (28)$$

where $\beta_D \ll 1$. The reason for having the squared β_D term for time is that within the governing equation of flow, the temporal derivative is first-order whereas the spatial derivative is second-order. The quantities τ and R are auxiliary variables as defined above.

The outer limit process of ϕ_D is denoted as ϕ_0 . The inner limit processes of ϕ_D

169 and ϕ_{wD} are denoted as ϕ_0^* and ϕ_{w0}^* respectively. The solution for the outer limit
 170 process is (Theis, 1935)

$$\phi_0 = B \cdot \text{Ei} \left(\frac{R^2}{4\tau} \right) \quad (29)$$

171 where B is an integration constant yet to be defined and Ei denotes the expo-
 172 nential integral.

173 For the inner region near the abstraction well it is better to revert back to the
 174 variable r_D such that the inner limit process is characterized by (recall equations
 175 9 and 12)

$$\beta_D^2 \frac{\partial \phi_0^*}{\partial \tau} + \frac{1}{r_D} \frac{\partial}{\partial r_D} (r_D q_0^*) = 0 \quad (30)$$

$$\beta_D^2 \frac{r_{cD}^2}{2} \frac{d\phi_{w0}^*}{d\tau} - 1 + r_{wD} q_0^*(r_D = r_{wD}) = 0 \quad (31)$$

176 where q_0^* satisfies $q_0^* + \beta_D q_0^{*2} = -d\phi_0^*/dr_D$. When $\beta_D \ll 1$, equation (30) in
 177 conjunction with equation (31) has the analytical solution

$$\phi_0^* = \frac{\beta_D}{r_D} - \ln(r_D) + C + O(\beta_D^2) \quad (32)$$

178 where C is another integration constant. Note that it is possible for C to be a
 179 function of τ (Roose et al., 2001).

180 The constants B and C are determined by matching the inner and outer limit
 181 processes, i.e.

$$\lim_{r_D \rightarrow \infty} \phi_0^* = \lim_{R \rightarrow 0} \phi_0 \quad (33)$$

182 For small R , equation (29) can be expanded to get (Cooper and Jacob, 1946)

$$\phi_0 = B[0.5772 + 2\ln(r_D) + 2\ln(\beta_D) - \ln(4\tau)] + O(\beta_D^2) \quad (34)$$

183 Therefore by applying equation (33), it can be seen that

$$B = -\frac{1}{2}, \quad C = -\frac{1}{2}[0.5772 + 2\ln(\beta_D) - \ln(4\tau)] \quad (35)$$

184 Adding the inner and outer limits and subtracting out of their sum the term that
 185 is common to both expressions in the overlap domain then yields the composite
 186 solution

$$\tilde{\phi}_0 = \frac{1}{2} \left[\ln \left(\frac{4t_D}{r_D^2} \right) - 0.5772 \right] + \frac{\beta_D}{r_D} \quad (36)$$

187 The mathematical development above provides a more rigorous derivation for
 188 the large time approximation proposed by Kelkar (2000) and Wu (2002a). Figure
 189 2 verifies that both the finite difference solution and the Sen (1988) solution, given
 190 in equation (27), correctly converge on to the large time approximation given in
 191 equation (36). However, it is still unclear which solution is more accurate at small
 192 and intermediate times.

193 **7 Laplace transform solution for large β_D**

194 In this section we follow the linearization procedure used by Odeh and Yang
 195 (1979), Ikoku and Ramey (1979) and Wen et al. (2007) to look at non-Darcy flow
 196 problems using the Izbash equation. The starting point is to rearrange equation
 197 (9) to get

$$q_D \frac{\partial \phi_D}{\partial t} + \frac{q_D^2}{r_D} + \frac{1}{2} \frac{\partial^2 q_D^2}{\partial r_D^2} = 0 \quad (37)$$

198 When β_D is very large, equation (10) reduces to

$$q_D^2 = -\frac{1}{\beta_D} \frac{\partial \phi_D}{\partial r_D} \quad (38)$$

199 which on substitution into equation (37) yields

$$q_D \frac{\partial \phi_D}{\partial t} - \frac{1}{\beta_D r_D} \frac{\partial \phi_D}{\partial r_D} - \frac{1}{2\beta_D} \frac{\partial^2 \phi_D}{\partial r_D^2} = 0 \quad (39)$$

200 To linearize the above equation, it is assumed that the q_D term on the left-
 201 hand-side is approximately r_D^{-1} (Odeh and Yang, 1979; Ikoku and Ramey, 1979;
 202 Wen et al., 2007), which certainly becomes true at very large times (Chen and Liu,
 203 1991). Applying the Laplace transform

$$\hat{\phi}_D(p) = \int_0^\infty \phi_D(t_D) e^{-pt_D} dt_D \quad (40)$$

204 then leads to the linear ordinary differential equation (assuming a zero initial
 205 condition)

$$\beta_D p \hat{\phi}_D - \frac{d\hat{\phi}_D}{dr_D} - \frac{r_D}{2} \frac{d^2 \hat{\phi}_D}{dr_D^2} = 0 \quad (41)$$

206 subjected to

$$\begin{aligned} \hat{\phi}_D &= \hat{\phi}_{wD}, & r_D &= r_{wD} \\ \hat{\phi}_D &= 0, & r_D &= \infty \end{aligned} \quad (42)$$

207 which has the analytical solution

$$\hat{\phi}_D = \hat{\phi}_{wD} \left(\frac{r_{wD}}{r_D} \right)^{1/2} \frac{K_1[(8p\beta_D r_D)^{1/2}]}{K_1[(8p\beta_D r_{wD})^{1/2}]} \quad (43)$$

where K_n denotes an n th order modified Bessel function of the second kind.

To obtain an expression for the Laplace transform of the well-bore head, $\hat{\phi}_{wD}$,

equation (38) must first be substituted into equation (12) to get

$$\frac{r_{cD}^2}{2} \frac{d\phi_{wD}}{dt_D} - 1 - r_{wD} \left[\frac{1}{\beta_D q_D} \frac{\partial \phi_D}{\partial r_D} \right]_{r_D=r_{wD}} = 0 \quad (44)$$

To linearize the above equation the remaining q_D term is again assumed to be approximately r_D^{-1} . Applying the Laplace transform and the initial condition in equation (13) then leads to the linear ordinary differential equation

$$p \frac{r_{cD}^2}{2} \hat{\phi}_{wD} - \frac{1}{p} - \frac{r_{wD}^2}{\beta_D} \frac{d\hat{\phi}_D}{dr_D} \Big|_{r_D=r_{wD}} = 0 \quad (45)$$

Differentiating equation (43) and substituting into equation (45) then yields

$$\hat{\phi}_{wD} = \frac{p^{-1} \beta_D K_1(x)}{(r_{wD} + 0.5 r_{cD}^2 p \beta_D) K_1(x) + r_{wD}^{3/2} (2p\beta_D)^{1/2} K_0(x)} \quad (46)$$

where $x^2 = 8p\beta_D r_{wD}$.

Furthermore, it can be shown that for an infinitesimal well

$$\lim_{r_{wD} \rightarrow 0} \hat{\phi}_D = \left(\frac{8\beta_D^3}{pr_D} \right)^{1/2} K_1[(8p\beta_D r_D)^{1/2}] \quad (47)$$

Equations (43), (46) and (47) represent special cases of the more general problem solved by Wen et al. (2007) who considered the Izbash (1931) equation, $q_D^n = -d\phi_D/dr_D$, where $n > 0$.

Figure 3 compares the Laplace transform solution for the well-bore head given in equation (46) with the finite difference solution. For all simulations both r_{wD} and r_{cD} were set to 1. The Laplace transform solution was inverted numerically using the de Hoog et al. (1982) algorithm. It can be seen that for very turbulent flow (i.e. $\beta_D/r_D > 10^3$), the correspondence between the Laplace transform and the finite difference solution is very good. It is also interesting to note that the finite difference solution results also correspond excellently with those presented by Wu (2002b) in his Figure 7. However, the good correspondence between the Laplace transform solution and the finite difference solution at small times is largely due to the well-bore storage dominating the head response. As shown in the next section, far away from the abstraction well, the small time response of the Laplace transform solution becomes inaccurate due to the linearization procedure. Nevertheless, this exercise builds more confidence into the accuracy of the small and intermediate time response of the finite difference solutions presented in this paper and by Wu (2002b).

8 Similarity solution for large β_D

Interestingly, it can be shown that for large β_D , the Sen (1988) solution (equations 26 and 27) reduces to

$$\frac{r_D}{\beta_D} \phi_D = 1 + \frac{\zeta}{\zeta + 2} + \zeta \cdot \ln \left(\frac{\zeta}{\zeta + 2} \right), \quad \zeta = \frac{\beta_D r_D}{t_D} \quad (48)$$

Figure 4 compares the finite difference solution with the Sen (1988) solution given in equation (27) for various values of β_D/r_D , with the axes transformed to

240 emphasize large β_D/r_D behavior. As with equation (48), it can be seen that for
 241 $\beta_D/r_D > 10^3$ the finite difference solution also converges to a single curve.

242 Indeed, for the case of the infinitesimal well and large β_D a similarity solution
 243 does exist. Applying the DVT (dependent variable transform), $u = r_D\phi_D/\beta_D$ and
 244 the IVT, $\zeta = \beta_D r_D/t_D$, the problem defined by equations (9) and (38) reduces to
 245 the non-linear ordinary differential equation

$$\zeta \frac{du}{d\zeta} = \frac{d}{d\zeta} \left[\left(u - \zeta \frac{du}{d\zeta} \right)^{1/2} \right] \quad (49)$$

246 The boundary condition at $r_D = \infty$ is satisfied by the form of the DVT. The
 247 boundary condition at $r_D = 0$ and the initial condition are transformed by the IVT
 248 to (recall equation 24):

$$\begin{aligned} u - \zeta \frac{du}{d\zeta} &= 1, & \zeta &= 0 \\ u &= 1, & \zeta &= \infty \end{aligned} \quad (50)$$

249 Unfortunately, equation (49) is still highly non-linear and therefore difficult to
 250 solve. However, Figure 5 shows the finite difference solution, when $\beta_D/r_D > 10^3$
 251 and $r_D \gg r_{wD}$. Note that the x-axis in Figure 5 is ζ whereas in Figure 4 it is
 252 equivalent to ζ^{-1} . It can be seen that, according to the finite difference solution,
 253 the solution to equations (49) and (50) should have a log-log slope of around -2
 254 for large ζ and ultimately should equal 1 when $\zeta \rightarrow 0$. There are several possible
 255 functions, $u_a(\zeta)$ that satisfy this criteria. However, after many exercises studying
 256 different functions, it was found that

$$u_a = (1 + \zeta)^{-2} \quad (51)$$

257 was the best choice. A similar approach was adopted by Lockington (1997) to
 258 obtain approximate solutions to the Boussinesq equation.

259 Figure 5 also compares the special case of the Sen (1988) solution given in
 260 equation (48), the Laplace transform solution (when $r_w \rightarrow 0$) as given in equation
 261 (47) and the proposed function given in equation (51). As seen in previous plots
 262 it is found that the Sen (1988) solution overestimates the finite difference solu-
 263 tion during small times, underestimates it during intermediate times but performs
 264 well during large times. The Laplace transform solution converges onto the finite
 265 difference solution faster than Sen's solution although during small times it is un-
 266 derestimating considerably. This is due to its associated linearization procedure.
 267 The proposed function in equation (51), although not exact, accurately follows the
 268 finite difference solution during all times.

269 It can also be shown that equation (51) is a more accurate solution to the
 270 similarity problem described by equations (49) and (50) than the special case of
 271 Sen's solution given in equation (48). Because both equations (48) and equation
 272 (51) satisfy the boundary conditions in equation (50) exactly, it is necessary only
 273 to focus on equation (49). The error, ϵ_a associated with using an approximate
 274 solution $u_a(\zeta) \approx u(\zeta)$ can be quantified by (recall equation 49)

$$\epsilon_a = \zeta \frac{du_a}{d\zeta} - \frac{d}{d\zeta} \left[\left(u_a - \zeta \frac{du_a}{d\zeta} \right)^{1/2} \right] \quad (52)$$

275 Substituting equation (48) into equation (52) yields

$$\epsilon_a = 1 + \frac{\zeta}{\zeta + 2} + \zeta \cdot \ln \left(\frac{\zeta}{\zeta + 2} \right) - \frac{2}{(\zeta + 2)^2} \quad (53)$$

276 whereas substituting equation (51) into equation (52) yields

$$\varepsilon_a = \frac{1}{(1 + \zeta)^2} \left[\frac{3\zeta}{(1 + 4\zeta + 3\zeta^2)^{1/2}} - \frac{2\zeta}{(1 + \zeta)} \right] \quad (54)$$

Figure 6 compares the error, ε_a associated with Sen's solution and the proposed function $(1 + \zeta)^{-2}$ using equations (53) and (54) respectively. It is clear that $(1 + \zeta)^{-2}$ consistently provides a better approximation to the true problem.

9 Conclusions

The derivation of Sen's similarity solution has been examined in detail. Unfortunately, it fails to satisfy the crucial requirement that the initial and boundary conditions be reducible to similarity form (e.g. Kevorkian, 1990, p.8). Consequently, the concern of Camacho-V. and Vasquez-C. (1992), that Sen's formula (equation (27)) is not a true similarity solution, is valid. Nevertheless, Sen's formula correctly converges on to the large time approximation of Kelkar (2000) and Wu (2002a) (see Figure 2), which we have rigorously derived using the method of matched asymptotic expansion (see section 6). Furthermore, Sen's solution becomes an approximate similarity solution with the correct variable combinations for large flow rates ($\beta_D/r_D \gg 10^3$) (see Figure 4 and compare equations (48) and (49)). However, it was found to slightly overestimate at small times and slightly underestimate at intermediate times as compared to the finite difference solution (see Figures 1 and 5).

The method of Wen et al. (2007) was used to derive a new approximate Laplace transform solution (equations (43), (46) and (47)) for Forchheimer flow to a well, designed to work well for large times ($t_D \gg 1$) and large flow rates ($\beta_D/r_D >$

297 10^3). This was found to underestimate the finite difference solution considerably
 298 at small times but became increasingly accurate at large times (see Figure 5). Fur-
 299 thermore, the discrepancy at small times became unimportant within the well-bore
 300 due to the dominating effect of well-bore storage (see Figure 3).

301 For large flow rates ($\beta_D/r_D > 10^3$), far away from the abstraction well ($r_D \gg$
 302 r_{wD}) it was shown that the original problem of Forchheimer flow to a well col-
 303 lapses onto a similarity solution (see Figure 4 and equation (49)) which is accu-
 304 rately approximated by $u_a = (1 + \zeta)^{-2}$. A subsequent error analysis showed the
 305 aforementioned heuristic function to be significantly more accurate than the Sen
 306 (1988) solution when $\beta_D/r_D > 10^3$ (see Figure 6).

307 An exact solution for transient Forchheimer flow to a well does not currently
 308 exist. However, this paper has presented a number of approximate solutions that
 309 can be used to confidently verify a numerical model of transient Forchheimer flow
 310 to a well. At large times (i.e. $t_D \gg 1$) a numerical model should replicate the
 311 response provided by equation (36) (e.g. Figure 2). When the flow rate is very
 312 large (i.e. $\beta_D/r_{wD} > 10^3$) and the well-bore storage is significant (i.e. $r_{cD} >$
 313 r_{wD}), a numerical model should closely replicate the well-bore response given by
 314 equation (46) (e.g. Figure 3). When the flow rate is very large (i.e. $\beta_D/r_D > 10^3$)
 315 and the well radius can be assumed infinitesimally small (i.e. $r_D \gg r_{wD}$), a
 316 numerical model should closely replicate the heuristic function given in equation
 317 (51) (e.g. Figure 5). Obviously, when the flow rate is very small (i.e. $\beta_D \rightarrow 0$) a
 318 numerical model should accord with solutions associated with Darcian flow.

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10 Notation

K	hydraulic conductivity [LT^{-1}];
m	aquifer thickness [L];
p	Laplace transform variable [-];
q	water flux [LT^{-1}];
Q	abstraction rate [L^3T^{-1}];
r	radial distance [L];
r_c	radius of well casing [L];
r_w	well radius [L];
S_s	specific storage coefficient [L^{-1}];
t	time [T];
β	turbulent flow coefficient [L^{-1}T];
ϕ	hydraulic head [L];
ϕ_w	hydraulic head in the well-bore [L];
$q_D = -2\pi m^2 q / Q$	dimensionless water flux;
$r_D = r / m$	dimensionless radius;
$r_{cD} = r_c / (S_s^{1/2} m^{3/2})$	dimensionless radius of well casing;
$r_{wD} = r_w / m$	dimensionless well radius;
$t_D = Kt / (S_s m^2)$	dimensionless time;
$\beta_D = -Q\beta / (2\pi m^2)$	dimensionless turbulent flow coefficient;
$\phi_D = -2\pi m K \phi / Q$	dimensionless hydraulic head;
$\phi_{wD} = -2\pi m K \phi_w / Q$	dimensionless hydraulic head in the well-bore;
$R = \beta_D r_D$	stretched dimensionless radius;
$u = r_D \phi_D / \beta_D$	dependant variable transform;
$\zeta = \beta_D r_D / t_D$	independent variable transform;
$\xi = r_D^2 / t_D$	independent variable transform;
$\tau = \beta_D^2 t_D$	stretched dimensionless time;

Figure 1: Comparison of the finite difference solution with the Sen (1988) solution given in equation (27) for various values of β_D / r_D .

Figure 2: Comparison of the finite difference solution, the Sen (1988) solution given in equation (27) and the large time approximation given in equation (36) for various values of β_D / r_D .

Figure 3: Comparison of the finite difference solution with the Laplace transform solution for the well-bore head given in equation (46) for various values of β_D / r_{wD} .

Figure 4: Comparison of the finite difference solution with the Sen (1988) solution given in equation (27) for various values of β_D / r_D , with the axes transformed to emphasize the large β_D / r_D behavior.

Figure 5: Comparison of the finite difference solution (when $\beta_D/r_D > 10^3$ and $r_D \gg r_{wD}$), the special case of the Sen (1988) solution given in equation (48), the Laplace transform solution (when $r_w \rightarrow 0$) as given in equation (47) and the proposed function given in equation (51).

Figure 6: Error comparison of the Sen (1988) solution and the proposed function $(1 + \zeta)^{-2}$ using equations (53) and (54) respectively.