A micromechanical model for kink–band formation: Part II — analytical modelling

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Abstract

An analytical micromechanical model for kink–band formation in an unidirectional fibre-reinforced composite is developed. This is supported by the conclusions of experimental and numerical programmes (presented in *Part I* of this paper) and is based on the equilibrium of an imperfect fibre laterally supported by an elasto–plastic matrix. The model predicts the longitudinal compressive strength of the composite (in closed form), the deflection and main stress fields in fibres and matrix at different stages of kink–band formation, the kink–band width, and the orientation of the fibres at the onset of fibre failure.

Key words: B. Kinking, C. Micro-mechanics, C. Analytical modelling, A. Carbon fibres

1 Introduction

Kink–band formation (figure 1.a) is the most common failure mode of fibre reinforced polymer (FRP) composites under longitudinal compression [2]. In this paper, an analytical micromechanical model for fibre kinking, based on the results of previous experimental and numerical investigations (*Part I* of this paper [3]), is derived.

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(a) Micrograph from experiments in CFRP [1] and kink-band geometric parameters: fibre angle α , band angle β , and band width w .

(b) Equilibrium of an infinitesimal part of the fibre.

Fig. 1. Kink-band.

Composite materials are preferred for advanced structural applications due to their high specific mechanical properties. To fully exploit these advantageous properties, it is necessary to accurately model the structural response of these materials. However, despite the large amount of work published on fibre kinking, there is no universal agreement among researchers on the micromechanics of this process and the appropriate modelling approach [2]. More recently, the results from the first World-Wide Failure Exercise [4] — where 19 failure theories for fibre reinforced polymer composites were applied to 14 different problems — showed that even the 5 top ranked theories have significant weaknesses; many of these are related to load cases where fibre kinking is likely to occur (longitudinal compression and shear).

Two approaches — microbuckling and kinking — are found in the literature to explain kink-band formation in FRPs [2]. It is generally accepted that they differ in the sense that the kinking approaches are not based on an instability at the micro-scale, but rather on matrix yielding or cracking due to initial fibre misalignments and further phase-shifted ($\beta \neq 0$) rotation due to loading.

Rosen [5] proposed the first model for FRP failure under longitudinal compression; his analysis is based on the in-phase microbuckling of a 2D layered (fibres and matrix) medium, elastic and with no initial imperfection. The composite longitudinal compressive strength X_C^C is predicted as

$$X_C^C = \frac{G_m}{1 - V_f}, \quad (1)$$

where G_m is the matrix shear modulus and V_f the fibre volume fraction in the composite. This model, and many of its later refinements, overestimate considerably the experimental values found for the composite compressive strength.

It was only with Argon [6] that the quantitative agreement of the predictions started improving significantly. His analysis was the first one considering kinking to be independent from microbuckling. He proposed that, during compression, an initial local fibre misalignment (θ_0) would induce fibre bending and matrix shearing, forcing the material to rotate further in a positive feedback process, and leading subsequently to ultimate failure. The longitudinal compressive strength was then found to be defined at the moment when the shear stresses in a coordinate system aligned with the initially misaligned fibres reach the matrix shear yield strength (S_m):

$$X_C^C = \frac{S_m}{\theta_0} . \quad (2)$$

Argon's model does not take into account the additional (to the initial misalignment) fibre rotation that occurs during longitudinal compression.

After Argon, several other researchers proposed models for kinking under similar principles, taking into account the effect of initial fibre misalignments and matrix yielding in shear. Hahn and Williams [7], Steif [8], Fleck et al [9,10] and Morais and Marques [11] presented different models for the process, all having in common the analysis of a geometrically imperfect (sinusoidal) fibre or smeared composite in bending (figure 1.b), under the action of an eccentric compressive load and shear stresses transmitted by the surrounding material with a non-linear constitutive response.

Hahn and Williams' model [7] predicts the composite compressive strength as

$$X_C^C = V_f \cdot G_C \cdot \frac{\gamma_{critical}}{\gamma_{critical} + \pi \cdot \frac{\overline{y_0}}{L}} , \quad (3)$$

where $\gamma_{critical}$ is the average shear strain in the composite that maximizes the function $X_C^C(\gamma_{critical})$, $G_C(\gamma) = G_m / (1 - V_f)$ is the composite shear tangent modulus for that strain, and $\overline{y_0}$ and L are respectively the initial imperfection amplitude and wave length. The correlation between Hahn's model and his experimental data is good, especially for composites with stiff matrix, but his analysis is concerned with the composite longitudinal strength only and not with the softening process that takes place after the peak load is reached.

Steif [8] considers that fibre failure occurs when the maximum tensile strain (combining the effect of compression and bending) reaches the fibre tensile fracture strain. Steif's model is able to reproduce the load versus displacement response during kinking, predicting the composite compressive strength and fibre failure. However, the kink-band width is an input parameter to the model, as the two points of maximum curvature in the fibre (at which failure occurs) are kept at a

distance equal to the initial fibre half-wavelength L .

Fleck et al [9] proposed a *bending theory* for fibre kinking, which is (together with its further developments [10]) widely recognized. The authors assume an initial sinusoidal fibre imperfection, forming an initial misaligned band inclined at an angle β ; fibres and matrix are smeared in a homogeneous anisotropic solid. The material is in equilibrium under compressive stresses, bending moments, and shear stresses; the corresponding differential equilibrium equation is solved numerically, giving as outputs the deformed shape of the material. The kink-band width (computed in the longitudinal direction only) is defined at the onset of fibre *tensile* failure due to local bending and remote compression. The composite compressive strength is found to be highly dependent of the amplitude of the imperfection, while the kink-band width is fairly insensitive to that parameter.

Morais and Marques [11] model the behaviour of a fibre during kinking by solving the governing differential equation using load incrementation, until the composite strength is reached. Their study is focused on the sensitivity analysis of the effect of material properties and geometric parameters on the composite strength. The model imposes directly a sinusoidal and in-phase deformed shape with constant wavelength, and requires an iterative process to compute the composite strength.

The model proposed in this paper is based on the equilibrium of an imperfect fibre loaded in compression and bending, and supported in shear by an yielding matrix. The model does not involve microbuckling, it imposes no constraints to the fibre deformed shape and it is able to reproduce the composite response from the beginning of compression until the onset of fibre failure. The paper is organised as follows: section 2 details the development of the analytical model; the results of a numerical application are presented and compared with the outputs from previous FE analyses in section 3; finally, in section 4 several implications of the model are discussed, and in section 5 the main conclusions summarized.

2 Development of the analytical micromechanical model

2.1 Strategy

The analytical model is formulated through the bending equilibrium of a fibre; it is based on an initial fibre misalignment and on matrix yielding in shear, which (according to the outputs from the experimental and numerical programmes pre-

Fig. 2. Schematics of the fibre considered in the model: geometry and loads.

sented in *Part I* of this paper [3]) plays an important role in the formation of a kink-band.

It is assumed that the fibres behave according to simple beam theory combined with compression. The matrix resists only to shear stresses; these are considered constant within each matrix layer across the radial direction, as the composite fibre content is typically high and thus the matrix layers thin [7,8,11]. The model requires mechanical continuity between matrix and fibres, as shear stresses are transferred through their interface. Geometrically, a 2D approximation of the real composite is used. A localized imperfection is assumed in the initially unloaded fibres; away from this, the fibres are kept straight throughout the analysis. Constitutively, a linear-elastic – perfect-plastic matrix behaviour in shear is assumed, so two domains — the *elastic domain* and the *softening domain* — are defined in the process. The initial imperfection amplitude, deflections and rotations are assumed to be small.

The composite behaviour is modelled from the beginning of compression until the onset of fibre failure; no further assumptions — notably on the deformed shape of the fibres and on the kink-band width — are introduced.

2.2 Fibre geometry and loading

The governing equations of this analytical model are deduced from the equilibrium of one single fibre within the composite, represented in figure 2. The fibre is loaded by the compressive force P , the bending moments M_B applied at its boundaries and the shear stresses transmitted by the matrix $\tau(x)$ acting on its surface [7,8,11]. The fibre initial misalignment is

$$y_0(x) = \overline{y}_0 \cdot \left(1 - \cos \left(\pi \cdot \frac{x}{L} \right) \right), \quad (4)$$

and the corresponding slope is denoted $\theta_0(x)$. The fibre undergoes a deflection $v(x)$, with associated slope $\theta(x)$. Its final position is $y(x)$, with slope $\theta_f(x)$.

2.3 2D equivalent model

For the sake of simplicity, an 2D layered representation of the composite with unit thickness (in the z-direction) is used. The fibres (perfect cylinders with diameter ϕ_f) are assumed to be arranged in a hexagonal pattern, being V_f the composite fibre volume fraction; a 2D equivalent model (figure 3.a) is taken from one of the symmetry planes, resulting in the width of the matrix layers between fibres

$$t_m = \phi_f \left(\sqrt{\frac{\pi}{2 \cdot \sqrt{3} \cdot V_f}} - 1 \right) . \quad (5)$$

2.4 Equilibrium of the fibre

The equilibrium of moments of an infinitesimal part of the fibre length (figure 1.b) results in:

$$\delta M + P \cdot \delta y - \tau \cdot \phi_f \cdot \delta s = 0 . \quad (6)$$

In equation 6, the bending moment M per unit thickness is given by the bending theory for thin and straight fibres and small deflections,

$$M = E_f \cdot I_f \cdot \frac{d^2 v}{dx^2} , \quad (7)$$

where E_f and I_f are respectively the fibre Young's modulus and second moment of area per unit thickness.

The term corresponding to the compressive load P per unit thickness can be expressed as

$$P \cdot \delta y(x) = P \cdot \delta y_0(x) + P \cdot \delta v(x) . \quad (8)$$

2.5 Shear stresses acting on the fibres

According to the hypothesis of thin layers of matrix, the shear traction acting on the fibre surfaces $\tau(x)$ can be obtained from the average shear stresses developed within the matrix.

Shear stresses in the matrix $\tau_{12}^m(x)$ are defined by its deformed geometry and constitutive law:

Fig. 3. 2D equivalent and matrix deformation.

(i) In the elastic domain,

$$\tau_{12}^m(x) = G_m \cdot \gamma_{12}^m(x), \quad (9)$$

where G_m is the shear modulus of the matrix. The fibres rotation here is very small, so the deformed shapes of adjacent fibres are approximately in-phase, as indicated in figure 3.b (as suggested by Morais and Marques [11]); the shear deformation of each matrix layer is then defined as

$$\gamma_{12}^m = \frac{\delta v^m}{\delta x} + \frac{\delta u^m}{\delta y} = \frac{\delta x \cdot \tan \theta}{\delta x} + \frac{\phi_f \cdot \tan \theta}{\delta y}. \quad (10)$$

As the fibre rotation $\theta(x)$ is small and the layer of matrix is thin (such as $\delta y \approx t_m$), then equation 10 results in

$$\gamma_{12}^m(x) = \left(1 + \frac{\phi_f}{t_m}\right) \theta(x). \quad (11)$$

For a 2D layered composite, the fibre volume fraction is given by equation 12.a, and an equivalent shear modulus can be defined by equation 12.b.

$$V_f^{2D} = \frac{\phi_f}{\phi_f + t_m} \quad (a) \quad G_m^{2D} = \frac{G_m}{1 - V_f^{2D}} \quad (b) \quad (12)$$

Using equation 12.a to eliminate ϕ_f/t_m in equation 11 gives

$$\gamma_{12}^m(x) = \frac{1}{1 - V_f^{2D}} \cdot \theta(x), \quad (13)$$

and using equations 12.b and 13 in equation 9 then the shear stresses in the matrix in the elastic domain can be expressed (as suggested by Steif [8]) as

$$\tau_{12}^{m, elastic}(x) = G_m^{2D} \cdot \theta(x). \quad (14)$$

- (ii) In the plastic domain, $\tau_{12}^{m, plastic}(x) = S_m$, where S_m is the yield strength of the matrix in shear.

Summarizing, the shear tractions acting on the fibre surfaces are

$$\tau(x) = \begin{cases} G_m^{2D} \cdot \theta(x) & , \text{ if } \theta(x) \leq \frac{S_m}{G_m^{2D}} \\ S_m & , \text{ if } \theta(x) > \frac{S_m}{G_m^{2D}} \end{cases} . \quad (15)$$

2.6 Governing differential equations

Substituting equations 7, 8 and 15 in the equilibrium equation 6, considering θ to be small (so $\delta s \approx \delta x$), and using $\theta(x) = \delta v(x)/\delta x$, then the governing differential equations of the problem become:

- (i) Before matrix yielding (*pre-yielding*):

$$E_f \cdot I_f \cdot \frac{d^2 \theta^{pre}(x)}{dx^2} - [G_m^{2D} \cdot \phi_f - P] \cdot \theta^{pre}(x) = -P \cdot \frac{dy_0(x)}{dx} , \quad (16)$$

where $\theta^{pre}(x) = \theta(x) \leq \frac{S_m}{G_m^{2D}} ;$

- (ii) Where the matrix has yielded (*post-yielding*):

$$E_f \cdot I_f \cdot \frac{d^2 \theta^{post}(x)}{dx^2} + P \cdot \theta^{post}(x) = -P \cdot \frac{dy_0(x)}{dx} + \phi_f \cdot S_m , \quad (17)$$

where $\theta^{post}(x) = \theta(x) > \frac{S_m}{G_m^{2D}} .$

2.7 Continuity and boundary conditions

The continuity and boundary conditions for equations 16 and 17 are defined differently for the two domains of compression:

- (I) In the *elastic domain* (figure 4.a), the matrix follows a linear-elastic behaviour within the entire length L ; the equation for *pre-yielding* (equation 16) applies to the whole domain in $0 \leq x \leq L$, under the following boundary conditions:
- $\theta^{pre}(0) = \theta^{pre}(L) = 0$, to avoid rotation at the boundaries;

Fig. 4. Boundary and continuity conditions.

- $v^{pre}(0) = 0$, to avoid rigid body movement.
- (II) In the *softening domain* (figure 4.b), the fibre rotations $\theta(x)$ in the *yield band* ($a \leq x \leq L - a$) are large enough to promote matrix yielding, so the equation for *post-yielding* (equation 17) applies; in the *elastic regions* ($0 \leq x \leq a$ or $L - a \leq x \leq L$), the equation for *pre-yielding* (equation 16) is still valid. The boundary conditions are:
 - $\theta^{pre}(0) = 0$, to avoid rotation at the boundary;
 - $v^{pre}(0) = 0$, to avoid rigid body movement;
 - $\theta^{post}(L/2) = 0$, to impose an anti-symmetric shape on the deflection.
 Continuity between the *elastic regions* and *yield band* is ensured at $x = a$ by:
 - $v^{pre}(a) = v^{post}(a)$, continuity of fibre deflection;
 - $\theta^{pre}(a) = \theta^{post}(a) = S_m/G_m^{2D}$, continuity of fibre slope and matrix shearing;
 - $\theta^{pre'}(a) = \theta^{post'}(a)$, continuity of curvature and bending moments in the fibre (additional condition, used to locate $x = a$).

2.8 Composite compressive strength

The (2D) composite compressive strength X_C^C is defined by the peak load per unit thickness P^{peak} as

$$X_C^C = P^{peak} \cdot \frac{V_f^{2D}}{A_f}, \quad (18)$$

where A_f is the area of the fibre cross section per unit thickness. It was found, from the numerical simulations previously performed, that the peak load occurs closely after the onset of matrix yielding (see the *Sequence of events* in *Part I* of this paper [3]), and is therefore estimated from:

$$\max \{\tau_{12}^m(x)\} = S_m. \quad (19)$$

Equations 16 (see resulting $\theta(x)$ in equation A.3) and 15 give, for the *elastic domain*, a maximum shear stress of

$$\max \left\{ \tau_{12}^{m,elastic}(x) \right\} = \tau_{12}^{m,elastic}(x = L/2) = G_m^{2D} \cdot \frac{\pi \cdot \frac{\overline{y_0}}{L} \cdot P}{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P}. \quad (20)$$

Introducing equation 20 into equation 19, and then the resulting peak load (as given in equation A.5) into equation 18, the composite compressive strength comes as

$$X_C^C = S_m \cdot \frac{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f}{S_m + \pi \cdot \frac{\overline{y_0}}{L} \cdot G_m^{2D}} \cdot \frac{V_f^{2D}}{A_f}. \quad (21)$$

2.9 Axial stresses and fibre failure

The axial stresses in the fibre $\sigma_{11}^f(x)$ result from the combination of the compressive load P and the bending moment $M(x)$:

- (a) Compressive load P . Assuming an uniform stress distribution over the fibre's cross section A_f and small slopes of the deformed shape (so $P \cdot \cos(\theta_0 + \theta) \approx P$), this stress is given by:

$$\sigma_{11}^{f,P} = \frac{P}{A_f}; \quad (22)$$

note that $\sigma_{11}^{f,P}$ is defined as being positive for compression.

- (b) Bending moment $M(x)$. From beam theory, the maximum axial stress within the fibre cross section is

$$\sigma_{11}^{f,M}(x) = \frac{M(x) \cdot \phi_f}{2 \cdot I_f}, \quad (23)$$

where $M(x)$ is given by equation 7; the value of x which maximizes $\sigma_{11}^{f,M}(x)$ is designated b , with $b \in [0, L/2]$.

Then, for a given load P and location $x \in [0, L/2]$, it comes (positive for compression):

$$\sigma_{11}^f(x, P) = \frac{P}{A_f} + \frac{\phi_f \cdot E_f}{2} \cdot \frac{d\theta(x)}{dx}. \quad (24)$$

Due to $\sigma_{11}^{f,P}$ being compressive and $\sigma_{11}^f(x, P)$ being symmetrical, the magnitude of the compressive axial stress in the fibre is always greater than that of the tensile one. In addition, experimental results presented in *Part I* of this paper [3] show that fibres are damaged first in compression. For these reasons, fibre failure

Table 1

Geometrical and composite parameters used in the analytical model.

L [μm]	$\overline{y}_0$ [μm]	V_f [%]	ϕ_f [μm]
750	15	60	7

Table 2

Constituents mechanical properties used in the analytical model.

E_f [GPa]	X_C^f [MPa]	G_m [GPa]	S_m [MPa]
276	3200	1.478	56

Table 3

Quantitative results from the analytical model at the peak load.

P^{peak} [N/mm]	$v^{peak}(L)$ [μm]	$\theta_f^{peak,average}$ [$^\circ$]	X_C^C [MPa]
5.62	3.37	2.51	654

(ff) initiates when the stress $\sigma_{11}^f(x, P)$ in equation 24 equals the fibre strength in compression X_C^f , for a load $P = P^{ff}$ and at a location $x = b^{ff}$. The condition is then

$$\frac{d\theta}{dx}(b^{ff}) = 2 \cdot \frac{X_C^f - \frac{P^{ff}}{A_f}}{\phi_f \cdot E_f}. \quad (25)$$

Considering damage propagation within the fibre cross section to be sudden, then the kink-band width can be roughly estimated by the location of onset of fibre failure ($b^{ff}, y(b^{ff})$) as

$$w \approx 2 \cdot \sqrt{[L/2 - b^{ff}]^2 + [y(L/2) - y(b^{ff})]^2}. \quad (26)$$

3 Results

The analytical model previously presented was solved and evaluated for successive loads P and using the input data from tables 1 and 2. Figures 5 and 6 show the analytical curves obtained, plotted together outputs of a previous numerical FE simulation (*Part I* of this paper [3]). The main quantitative results are provided in tables 3 and 4.

In the overall, the following results were obtained:

- The *load vs. displacement curve* (figure 5.a) shows an almost linear and stiff

(a) Load vs. transverse displacement curve.

(b) Shear stresses in the matrix.

(c) Fibre deflection.

(d) Axial stresses in the fibre (top surface).

Fig. 5. Results of the analytical model (compared to those of the *cohesive* model (detailed in *Part I* of this paper [3]), for a similar compressive load P).

Table 4

Quantitative results from the analytical model at the onset of fibre failure.

P^{ff} [N/mm]	$v^{ff}(L)$ [μm]	$\theta_f^{ff}(L/2)$ [°]	w [μm]
1.35	112.8	16.3	249

response in the *elastic domain* and, after the *peak load*, a progressively reducing load in the *softening domain*;

- The *resulting fields* for $v(x)$, $\theta(x)$, $\tau_{12}^m(x)$ and $\sigma_{11}^f(x)$ are, in the *elastic domain*, sinusoidal (see figures 5.b–d and 6.b–d, and corresponding equations in appendix

Fig. 6. Results of the analytical model (evolution of fields).

A.1). In the *softening domain*, these fields are much more complex (see figures 5.b–d and 6.b–d, and corresponding equations in appendix A.3) and with distinct shapes within the *elastic regions* and the *yield band*;

- The *shear stresses* in the matrix $\tau_{12}^m(x)$ (figures 5.b and 6.b) follow, in the *elastic domain*, a sinusoid with maximum at $x = L/2$. In the *softening domain*, a central *yield band* with constant stresses $\tau_{12}^m(x) = S_m$ is defined, outside which the shear stresses drop abruptly;
- The fibre *transverse displacements* $v(x)$ (figures 5.c and 6.c) in the *elastic domain*

are a magnification of the sinusoidal imperfection. In the *softening domain*, the deformed shape changes completely: the points of maximum curvature move suddenly inwards, away from the fixed ends of the model. As the compression progresses, the fibre ends become flatter relatively to the centre; the central inclined region (between the points of maximum curvature) rotates further and extends smoothly towards the fibre boundaries;

- The magnitude of the *axial stresses* in the fibre $\sigma_{11}^f(x)$ (figures 5.d and 6.d) in the *elastic domain* are maximum at the fibre boundaries and dominated by the compressive component $\sigma_{11}^{f,P}$. In the *softening domain*, the bending component $\sigma_{11}^{f,M}$ increases considerably inside the *yield band* (tensile stresses develop as well) and $\sigma_{11}^{f,P}$ is reduced;
- A matrix *yield band* forms almost instantaneously as the composite goes into the *softening domain*, and widens smoothly as compression progresses (figure 6.a). Also immediately after the peak load, the *maximum bending* points move from the fibre boundaries towards its centre, inside the *yield band*, and are kept at approximately half of the *yield band* width during further compression.

4 Discussion

4.1 Qualitative analysis of fields

The $P[v(x)]$ curve obtained with the analytical model (figure 5.a) represents the common behaviour documented for kink-band formation [12].

When it comes to the $v(x)$, $\theta(x)$, $\tau_{12}^m(x)$ and $\sigma_{11}^f(x)$ fields (equations in Appendix A and figure 5.b–d), in the *elastic domain* the response merely amplifies the initial imperfection; however, as soon as the matrix starts yielding, deformation localizes within the *yield band* and the shapes (and not only the amplitude) of the deflection and stress fields change completely, reflecting the typical response of kink-band formation [13].

In the elastic domain, the contribution of the matrix to the overall stiffness is very important (see for instance equation A.1 in Appendix A, where $G_m^{2D} \cdot \phi_f = 55.6\text{N/mm}$, against $\pi^2/L^2 \cdot E_f \cdot I_f = 0.14\text{N/mm}$ from fibres). When the *yield band* forms, the shear stress in the matrix is limited and therefore the material becomes softer there (figure 5.b). For this reason, fibre deflection increases significantly and locally, deforming into a kinked shape (figure 5.c). This type of deflection has an almost-zero curvature within the *elastic regions* and in the centre of the *yield band*;

near the *yield band* boundaries, the fibre is highly curved, which reflects on the $\sigma_{11}^f(x)$ field (figure 5.d).

4.2 Agreement between analytical and FE results

The analytical model previously developed is based on the outputs from the numerical simulations performed in *Part I* of this paper [3]; the analytical results are compared to the FE ones in figure 5. It is found that all the qualitative features present in the numerical curves are shown in the analytical ones, and the quantitative agreement is also remarkable. This proves that the numerical models were correctly interpreted, and that the main assumptions of the analytical model — regarding matrix deformation & constitutive law (section 2.5) and, consequently, the prescription of periodic boundary conditions in the *elastic regions* and plastic foundation in the *yield band* — accurately represent the behaviour during kink-band formation.

The $P[v(x)]$ analytical curve has a peak load below the numerical one (figure 5.a); this is due to the use of numerical damping in the FE routine and (as also noted by Kyriakides et al [14]) to the high number of matrix layers (through which the *yield band* must propagate transversely) in the numerical simulation. This effect is slowly reduced as compression proceeds, but has nevertheless a quantitative repercussion in the displacement $v(x)$ field (figure 5.c).

The onset of fibre failure in the numerical simulation starts in the outer fibres and progresses transversely (see FE results in *Part I* of this paper [3]). As the analytical model has no edge effect, it predicts fibre failure to start after the numerical onset of outer fibre failure. On the other hand, the curvature of the central fibre in the numerical model is reduced due to the transverse stiffness of the composite as a whole, delaying its failure. For these reasons, analytical onset of fibre failure occurs between the numerical onset of failure in outer and central fibres (figure 5.a).

4.3 Model outputs

The primary outputs of the analytical model are the deflection and stress fields (Appendix A); through them, the process of kink-band formation can be fully tracked until the onset of fibre failure. As no deformed shape for the compressed fibre is imposed a priori, the shape — and not only the magnitude — of these

fields are outputs of the model, and reproduce the expected behaviour of kink-band formation.

In addition, this model predicts the composite longitudinal compressive strength X_C^C (in closed form, equation 21), the kink-band width w and the orientation $\theta_f(x)$ of fibres at the onset of failure (which can be further related to the band angle β and fibre angle α). However, the behaviour after the onset of fibre failure needs to be studied further, to confirm the assumption of no effect of damage propagation in the kink-band width and orientation.

4.4 Sensitivity study

A sensitivity study was carried out, regarding the influence of the initial imperfection on the model outputs. The range of imperfections analysed was measured by Kyriakydes [14], who found regularly-spaced bands of highly-misaligned fibres with half-wavelength and half-amplitude of respectively $L \in [1050; 2800] \mu\text{m}$ and $\overline{y}_0 \in [21; 70] \mu\text{m}$. This range of imperfections is considerably large, with variations of nearly 50% relatively to the central values for both amplitude and wavelength; imperfection misalignments (measured by $\overline{y}_0/L$) ranged from 0.8 to 6.7%. The results from the sensitivity study are presented in figure 7.

For the whole range tested, the shape of stress & field curves and the sequence of events for kink-band formation are independent of the initial imperfection parameters.

The compressive strengths (figure 7.a) show a considerable dependency on the initial imperfection, with higher strengths verified for the less-pronounced misalignments (smaller amplitudes and larger wavelengths). This agrees with the well-known fact that the longitudinal compressive strength of UD CFRP is highly dependent on the manufacturing process and material defects.

Regarding the onset of fibre failure, the main parameters for kink-band formation — load, fibre-angle and kink-band width (figure 7.b-d) — evidence a reduced sensitivity to the imperfection parameters; for instance, the kink-band width is kept within $w \in [233; 369] \mu\text{m}$, which represents a variation of $\pm 22\%$ for misalignments varying almost 80% around the central value. Both the kink-band width and fibre angle increase for more severe imperfections, specially in terms of larger amplitudes (the effect of smaller wavelengths is slightly less pronounced).

Overall, the kink-band width and the fibre-angle at fibre failure are kept within a

Fig. 7. Sensitivity study for the initial imperfection.

reasonable range of values ($\theta_f^{ff}(L/2) \in [13; 24]^\circ$, i.e. $\pm 29\%$), comparable with the ones commonly observed experimentally [2,13]. In addition, these results show that the model here proposed is applicable to a wide range of imperfections, giving outputs and trends consistent with those obtained analytically by other researchers [9].

5 Conclusions

An analytical model has been developed for kink-band formation in compressively loaded unidirectional composites. This was based on the conclusions of experimental and numerical programmes (detailed in *Part I* of this paper [3]).

The model considers the equilibrium of a single imperfect fibre under compression and supported in shear by the matrix. Matrix yielding is found to be a key feature for kink-band formation.

As global outputs, the model is able to provide load vs. displacement curves and fields for the deflection, axial stresses in fibres and shear stresses in matrix, from the beginning of compression to the onset of fibre failure. It also predicts the composite compressive strength (in closed form), the kink-band width and the orientation of the fibres at the onset of their failure. A sensitivity study for the imperfection parameters showed the composite strength to be severely degraded by more pronounced imperfections; by opposition, the geometric parameters at the onset of fibre failure — kink-band width and fibre angle — were found to be kept within a narrow band of values, with good agreement with experimental data.

Unlike other analytical models for kink-band formation [8,11], no deformed shape is imposed a priori in the one here developed; all the features captured — namely the location of the *maximum bending* points inside the *yield band* and the deflection in a kinked (and not sinusoidal) shape — are also outputs of the model. The model is able to predict kink-band formation without fibre failure (width given in this case by the *yield band*), can be adapted (by modification of equations 14 and 15) to layered materials with frictional interface (like rocks), and can be extended to composites under hydrostatic pressure (by making the yield strength of the matrix a function of the hydrostatic pressure).

The results of the analytical model were compared against results from an FE model; a good agreement was found, both quantitatively and qualitatively.

6 Acknowledgments

The funding from University of Porto (Erasmus Programme) and Portuguese Foundation for Science and Technology (project POCTI/EME/50354/2003) is acknowledged. The Engineering and Physical Sciences Research Council, the UK Ministry

of Defence, Airbus and Renault F1 under the project EP/E0Z3169/1 are also gratefully acknowledged. The authors would like to thank Professor P.T. Curtis of the Defence Science and Technology Laboratory (DSTL) for many helpful discussions.

A Analytical expressions for displacement and stress fields

A.1 Elastic domain

- Transverse displacements

$$v^{pre}(x, P) = \frac{\bar{y}_0 \cdot P}{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P} \cdot \left(1 - \cos\left(\pi \frac{x}{L}\right)\right) \quad (\text{A.1})$$

- Slope

$$\theta^{pre}(x) = \frac{\pi \cdot \frac{\bar{y}_0}{L} \cdot P}{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P} \cdot \sin\left(\pi \frac{x}{L}\right) \quad (\text{A.2})$$

- Shear stresses

$$\tau_{12}^m(x, P) = G_m^{2D} \cdot \frac{\pi \cdot \frac{\bar{y}_0}{L} \cdot P}{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P} \cdot \sin\left(\pi \frac{x}{L}\right) \quad (\text{A.3})$$

- Axial stresses

$$\begin{aligned} \sigma_{11}^f(x, P) = & \frac{P}{A_f} + \\ & + G_m^{2D} \cdot \frac{\phi_f \cdot E_f}{2} \cdot \frac{\bar{y}_0 \cdot \frac{\pi^2}{L^2} \cdot P}{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P} \cdot \cos\left(\pi \frac{x}{L}\right) \end{aligned} \quad (\text{A.4})$$

A.2 Peak load

- Peak load

$$P^{peak} = S_m \cdot \frac{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f}{S_m + \pi \cdot \frac{\bar{y}_0}{L} \cdot G_m^{2D}} \quad (\text{A.5})$$

- Transverse displacements

$$v(x, P^{peak}) = \frac{L \cdot S_m}{\pi \cdot G_m^{2D}} \cdot \left(1 - \cos\left(\pi \frac{x}{L}\right)\right) \quad (\text{A.6})$$

A.3 Softening domain

- Slope in the elastic region

$$\begin{aligned} \theta^{pre}(x) = & C^{pre} \cdot \left(e^{\sqrt{\frac{G_m^{2D} \cdot \phi_f - P}{E_f \cdot I_f}} \cdot x} - e^{-\sqrt{\frac{G_m^{2D} \cdot \phi_f - P}{E_f \cdot I_f}} \cdot x} \right) + \\ & + \frac{P}{G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P} \cdot \frac{\pi \cdot \bar{y}_0}{L} \cdot \sin\left(\pi \frac{x}{L}\right), \quad x \leq a \end{aligned} \quad (A.7)$$

with

$$\begin{aligned} C^{pre} = & \frac{1}{e^{\sqrt{\frac{G_m^{2D} \cdot \phi_f - P}{E_f \cdot I_f}} \cdot a} - e^{-\sqrt{\frac{G_m^{2D} \cdot \phi_f - P}{E_f \cdot I_f}} \cdot a}} \cdot \\ & \cdot \left(\frac{S_m}{G_m^{2D}} - \frac{P}{\left(G_m^{2D} \cdot \phi_f + \frac{\pi^2}{L^2} \cdot E_f \cdot I_f - P\right)} \cdot \frac{\pi \cdot \bar{y}_0}{L} \cdot \sin\left(\pi \frac{a}{L}\right) \right) \end{aligned}$$

- Slope in the yield band

$$\begin{aligned} \theta^{post}(x) = & C_1^{post} \cdot \sin\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot x\right) + C_2^{post} \cdot \cos\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot x\right) + \\ & + \frac{S_m \cdot \phi_f}{P} - \frac{P}{P - \frac{\pi^2}{L^2} \cdot E_f \cdot I_f} \cdot \frac{\pi \cdot \bar{y}_0}{L} \cdot \sin\left(\pi \frac{x}{L}\right), \quad a < x \leq \frac{L}{2} \end{aligned} \quad (A.8)$$

with

$$\begin{aligned} C_1^{post} = & \frac{1}{\cot\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot \frac{L}{2}\right) \cdot \cos\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot a\right) + \sin\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot a\right)} \cdot \\ & \cdot \left(\frac{P \cdot \frac{\pi \cdot \bar{y}_0}{L} \cdot \sin\left(\pi \frac{a}{L}\right)}{P - \frac{\pi^2}{L^2} \cdot E_f \cdot I_f} - S_m \cdot \left(\frac{\phi_f}{P} + \frac{1}{G_m} \right) \right) \\ C_2^{post} = & \frac{1}{\cos\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot a\right) + \tan\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot \frac{L}{2}\right) \cdot \sin\left(\sqrt{\frac{P}{E_f \cdot I_f}} \cdot a\right)} \cdot \\ & \cdot \left(\frac{P \cdot \frac{\pi \cdot \bar{y}_0}{L} \cdot \sin\left(\pi \frac{a}{L}\right)}{P - \frac{\pi^2}{L^2} \cdot E_f \cdot I_f} - S_m \cdot \left(\frac{\phi_f}{P} + \frac{1}{G_m} \right) \right) \end{aligned}$$

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