ASYMPTOTICS FOR RAYLEIGH–BLOCH WAVES ALONG LATTICE LINE DEFECTS*

L. M. JOSEPH[†] AND R. V. CRASTER[†]

Abstract. High-frequency homogenization is applied herein to develop asymptotics for waves propagating along line defects in lattices; the approaches developed are anticipated to be of wide application to many other systems that exhibit surface waves created or directed by microstructure. With the aim being to create a long-scale continuum representation of the line defect that nonetheless accurately incorporates the microscale information, this development uses the microstructural information embedded within, potentially high-frequency, standing wave solutions. A two-scaled approach is utilized for a simple line defect and demonstrated versus exact solutions for quasi-periodic systems and versus numerical solutions for line defects that are themselves perturbed or altered. In particular, Rayleigh–Bloch states propagating along the line defect, and localized defect states, are identified both asymptotically and numerically. Additionally, numerical simulations of large-scale lattice systems illustrate the physics underlying the propagation of waves through the lattice at different frequencies.

Key words. homogenization, stop-bands, lattice materials, perfectly matched layers

AMS subject classifications. 35, 70

DOI. 10.1137/120872401

1. Introduction. Rayleigh–Bloch waves are a ubiquitous feature of periodic line defects in continuous, or discrete, media and arise in many apparently disconnected areas of engineering and physics: in water waves [14], in acoustic or electromagnetic gratings [30], elastic plates [15], on structured surfaces [17], in photonic lattices [32], among many other areas. In each case a periodic surface, or interface, is embedded within another bulk medium and a surface wave exists that propagates parallel to the surface but decays exponentially with distance from the surface. There has been considerable interest over the years in the existence of these waves, their properties, and finding at which frequencies these waves exist; there are also several subtleties involving finding whether embedded Rayleigh–Bloch waves exist [31]. It is worth noting that the facility to guide surface energy within a photonic crystal lattice [18] or along a surface [21] is a vital ingredient of modern applications such as designer surfaces in optics using surface plasmons. If one inserts a line interface or surface (not necessarily made of periodic cells) within a bulk material, then often a surface Rayleigh wave exists that also propagates along the surface and decays exponentially with distance from the surface; such waves are commonplace in elasticity [1] and also occur in other applications. Although the final effect is similar to Rayleigh–Bloch waves, the reason for their excitation is subtly different.

Much of the previous work on surface or Rayleigh–Bloch waves uses numerical, or in limited situations, approximate or analytical techniques, i.e., for comb-like surfaces [13, 14, 17], to detect and find such waves. The present article introduces an asymptotic technique aimed at the situation where one has a microstructured surface

*Received by the editors April 5, 2012; accepted for publication (in revised form) May 22, 2013; published electronically August 20, 2013.

<http://www.siam.org/journals/mms/11-3/87240.html>

[†]Department of Mathematics, Imperial College, London, SW7 2AZ, UK (l.joseph11@imperial.ac.uk, r.craster@imperial.ac.uk). The first author gratefully acknowledges an education grant from the A. G. Leventis Foundation. The second author acknowledges the EPSRC funding through grant EP/J009636/1.

or interface consisting of many thousands or even millions of periodic cells but where the overall structure is on a different lengthscale: this situation is common in real examples such as multilayer dielectric gratings for laser fusion [6], plasmonic solar cells [3, 26], and microwave absorbers [19]. The approach here is to develop a two-scale process in the simpler situation of line defects in discrete lattices using the model of Montroll and Potts [24] to illustrate and begin to develop the necessary mathematical theory of Rayleigh–Bloch and surface waves in phononic contexts.

The two-scale process we develop is an extension of homogenization theory; the usual multiple-scale procedure [5, 7, 23, 27, 33] as applied to multiscale media necessarily requires the wavelength to be long relative to microstructure, thereby limiting the theory to low frequencies; unfortunately, low frequencies are of little interest in many real applications. Indeed, for the situations considered in this article no surface waves have low frequency and the conventional theory is not applicable. Fortunately, for bulk media it has recently been understood how to develop a high-frequency homogenization for waves through doubly periodic or nearly periodic media in continua [11], frames [25], discrete media [10], and for elastic plates [2]. The usual asymptotic procedure [23] proceeds via the introduction of two scales: a slow scale $\mathbf{x}$ and a fast scale (microscopic) $\mathbf{y} = \mathbf{x}/\varepsilon$, where $\varepsilon \ll 1$. This scale separation shrinks the microstructure, allowing it to be isolated and averaged; the emphasis is upon the long scale. The frequency of the waves is forced to be effectively low and this limits the validity of the usual theory, with the leading-order displacement field eventually independent of the microscale. Craster, Kaplunov, and Pichugin [11] make a subtle change of emphasis beginning with a trivial change in the definition of the two scales with a slow scale $\mathbf{X} = \varepsilon\mathbf{x}$ and a fast scale $\boldsymbol{\xi} = \mathbf{x}$; this scaling fixes attention upon the microscale. The theory expounded in [11] then follows through the logic of this, together with an additional idea taken from physics: if one has a perfect lattice, then attention can be concentrated upon a simple cell with Bloch conditions set upon its edges; this perfect lattice problem encodes the multiple scattering between cells no matter how high the frequency is. Noting that there exist standing waves at the wavenumbers corresponding to the edges of the Brillouin zone, and that those standing waves have associated standing wave frequencies, inspires one to use this information within an asymptotic theory.

From almost time immemorial, discrete lattice structures have been used to model and interpret wave phenomena and have made a significant contribution to the development of wave mechanics [9]. There is not an exact one-to-one analogue problem in a continuous system. Initially, we consider a two-dimensional lattice structure constructed by rows of identical masses joined by massless strings with a single row of identical masses of different size, which can be regarded as a perfect line defect, as shown in Figure 1. We are also concerned with the case of a bulk lattice containing a line defect, which is itself defective by having a single mass different from the rest in this line. The aim is to find whether it is possible for a “Rayleigh–Bloch”-like wave to exist in such structures employing asymptotic and numerical methods. The two-scale asymptotic approach, which is implemented for solving the problem, uses microscale information to find asymptotic homogenized continuum equations valid on the macroscale. The results obtained from using this method are compared to those of numerical solutions.

As noted earlier we will concentrate here upon discrete lattice structures and use the discrete high-frequency homogenization theory of [12]. The article follows the following plan: we will consider line defects leading to Rayleigh–Bloch waves, in section 2, using exact techniques based upon discrete Fourier transforms and the asymptotic

FIG. 1. A perfect line defect embedded within a bulk lattice. (a) The bulk lattice consists of alternating strips of masses. (b) The bulk lattice consists of identical masses.

technique based around standing wave frequencies. In section 3, the same surface wave as in section 2 is reconsidered, but using two scales orientated perpendicular to the line defect; the two approaches are then compared. To demonstrate how these waves occur, and how the theory can be used to interpret the results, some illustrative examples from numerical simulations are included in section 4; the displacements are depicted in a discrete lattice with embedded lines of defective masses, excited at different frequencies. Finally, some concluding remarks are drawn together in section 5.

2. Perfect line defect. We begin with a line of identical masses, the line defect, that replace one line within an infinite square lattice; the line defect has masses that are different from those within the main lattice.

Before turning our attention to the line defect and the effect that it creates, it is worthwhile to outline the situation for the undisturbed main lattice constructed from alternating stripes of masses; typical dispersion curves are shown in Figure 2. For the infinite lattice it is convenient to fix upon an elementary cell of four masses arranged in a square (with the M_1 masses along the lower edge) and label the lower left mass as $y_{2n,2m}$. The displacements must satisfy the following difference equations:

$$(2.1) \quad y_{2n+1,2m} + y_{2n-1,2m} + y_{2n,2m+1} + y_{2n,2m-1} - 4y_{2n,2m} = -M_1\Omega^2 y_{2n,2m},$$

$$(2.2)$$

$$y_{2n+2,2m+1} + y_{2n,2m+1} + y_{2n+1,2m+2} + y_{2n+1,2m} - 4y_{2n+1,2m+1} = -M_2\Omega^2 y_{2n+1,2m+1},$$

$$(2.3)$$

$$y_{2n+2,2m} + y_{2n,2m} + y_{2n+1,2m+1} + y_{2n+1,2m-1} - 4y_{2n+1,2m} = -M_1\Omega^2 y_{2n+1,2m},$$

$$(2.4)$$

$$y_{2n+1,2m+1} + y_{2n-1,2m+1} + y_{2n,2m+2} + y_{2n,2m} - 4y_{2n,2m+1} = -M_2\Omega^2 y_{2n,2m+1}.$$

A similar arrangement is treated in [12], and the high-frequency homogenization method developed there creates asymptotic expressions for the dispersion curves in a two-dimensional lattice: it can similarly be used, if desired, for the cases shown in

FIG. 2. Dispersion curves for a perfect infinite square lattice. (a) A lattice of alternating strips of masses. (b) The reciprocal Brillouin lattice in wavenumber space κ_1, κ_2 (assuming a periodic square elementary cell of four masses). (c) The dispersion curves for $M_1 = 1, M_2 = 2$ (with masses M_1 as the lower row in the elementary cell). (d) The dispersion curves for $M_1 = M_2 = 1$. The dashed line in panel (d) depicts the dispersion curve starting at $(\kappa, \Omega) = (0, 2)$ in Figure 3, but here it is folded over in the middle; this occurs since the four-mass cell assumes periodicity $[0, \frac{\pi}{2}]$, unlike $[0, \pi]$, as used in Figure 3 for a single mass in a cell.

Figures 2(c,d); however, we do not do so here. Here we just show the exact dispersion curves, where $y_{2n+\hat{N}, 2m+\hat{M}} = \exp(i[\kappa_1 \hat{N} + \kappa_2 \hat{M}]) y_{2n, 2m}$ and (κ_1, κ_2) are the Bloch wavenumber vector κ with $|\kappa_{1,2}| \leq \pi/2$. The dispersion relation connecting frequency and wavenumber is found from an eigenvalue calculation as in [12]. In wavenumber space (Figure 2(b)) one need only consider the square shown, and plotting the dispersion curves around the triangle ABC' , or ADC' , suffices (from periodicity). We use the latter here, as the curves along AD correspond to waves propagating in the n direction, and later when we introduce a line defect along $m = 0$, we will draw connections with waves propagating along the defect. The resultant dispersion curves, in Figures 2(c,d) for dissimilar and identical masses, respectively, show that there is an infinite stop-band above a critical value ($\Omega^2 > (3(M_1 + M_2) + \sqrt{9(M_1^2 + M_2^2) - 14M_1M_2})/M_1M_2$ and $\Omega^2 > 8/M_1$, respectively) and, for dissimilar masses, Figure 2(c), a finite width stop-band. These dispersion curves show much that is of independent interest: a flat portion along AC in Figure 2 is of particular interest [4], as is the linear low-frequency behavior near the origin for which classical homogenization applies, and there are standing waves at the edges of the Brillouin zone, coalescences of dispersion

curves, and regions with negative group velocities. However, these are tangential to our purpose, which is to investigate the propagating modes that occur along a line defect introduced into this lattice.

To clearly illustrate the asymptotic technique for the surface waves, we specialize to identical masses in the bulk lattice for which

$$(2.5) \quad y_{n+1,m} + y_{n-1,m} + y_{n,m+1} + y_{n,m-1} - 4y_{n,m} = -\Omega^2 y_{n,m} \begin{cases} M & m \neq 0, \\ M_0 & m = 0. \end{cases}$$

Later when we consider whether Rayleigh–Bloch waves can travel along the line of different masses, we will again consider quasi-periodic Bloch conditions, but only in the x direction, and so the Bloch wavenumber will be a simple scalar κ .

Let us approach this by assuming that there are two scales in n ; that is, there is some long scale in which the discrete line defect becomes a continuum interface condition to the main bulk lattice, and a short scale that is simply one column of masses and their nearest neighbors (in n); for clarity we simply keep m discrete and do not rescale it. For the two scales in n we follow [12] and set $\xi = \varepsilon n$ as the long scale with $\varepsilon \ll 1$ and a short scale $N = (-1, 0, 1)$ and denote $y_{n,m}$ as

$$(2.6) \quad y_{n,m} = y_m(\xi, N).$$

The scale separation is set by ε and we shall leave this arbitrary, but assumed small, at present and then later fix its value for specific examples. The two variables ξ and N are now assumed independent, and we first consider the short scale. Here we have two choices: each column of masses can be assumed to be either in-phase or directly out-of-phase with its nearest neighboring column; we assume the former (and give results later for the latter). Thus on the microscale we assume that

$$(2.7) \quad y_m(\xi, 0) = y_m(\xi, \pm 1),$$

and so motion is locally periodic on the microscale. Then (2.5) in the two-scale notation is

$$(2.8) \quad \begin{aligned} & y_m(\xi + \varepsilon, 0) + y_m(\xi - \varepsilon, 0) + y_{m+1}(\xi, 0) + y_{m-1}(\xi, 0) - 4y_m(\xi, 0) \\ &= -\Omega^2 y_m(\xi, 0) \begin{cases} M & m \neq 0, \\ M_0 & m = 0. \end{cases} \end{aligned}$$

We now drop the 0 in the argument of y and consider it henceforth understood; following [12] we adopt the ansatz

$$(2.9) \quad y_m(\xi) = y_m^{(0)}(\xi) + \varepsilon y_m^{(1)}(\xi) + \varepsilon^2 y_m^{(2)}(\xi) + \cdots$$

and for the frequency

$$(2.10) \quad \Omega^2 = \Omega_0^2 + \varepsilon \Omega_1^2 + \varepsilon^2 \Omega_2^2 + \cdots.$$

Substitution of (2.9) and (2.10) in (2.8) gives a hierarchy of equations in power of ε that are then solved order-by-order in ε : at leading order of ε^0 ,

$$(2.11) \quad 2y_m^{(0)}(\xi) - y_{m+1}^{(0)}(\xi) - y_{m-1}^{(0)}(\xi) = \Omega_0^2 y_m^{(0)}(\xi) \begin{cases} M & m \neq 0, \\ M_0 & m = 0. \end{cases}$$

Equation (2.11) is independent of ξ , implying that $y_m^{(0)}(\xi) = f(\xi)Y_m$ where $f(\xi)$, the behavior of the solution on the long scale, is as yet unknown; indeed, the whole point is to find an asymptotic representation of this function $f(\xi)$. Importantly, $f(\xi)$ is defined only on the continuum macroscale and is independent directly of m, n , therefore giving a long-scale envelope of the behavior of $y_{n,m}$.

Using the discrete Fourier transform pair

$$(2.12) \quad \tilde{Y}(\alpha) = \sum_{m=-\infty}^{\infty} Y_m e^{im\alpha}, \quad Y_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{Y}(\alpha) e^{-im\alpha} d\alpha,$$

in (2.11) we obtain initially

$$(2.13) \quad \tilde{Y}(\alpha) = \frac{(M_0 - M)\Omega_0^2 Y_0}{2 - 2\cos\alpha - \Omega_0^2 M}.$$

Then by applying the inverse Fourier transform on (2.13) we deduce an expression for Y_m , and by setting $m = 0$ we derive an expression for Ω_0^2 :

$$(2.14) \quad Y_m = Y_0 \left(1 - \frac{\Omega_0^2 M_0}{2}\right)^m, \quad \Omega_0^2 = \frac{4M}{M_0(2M - M_0)}.$$

At first order in ε it transpires that $\Omega_1^2 = 0$ and that $y_m^{(1)}(\xi)$ can safely be ignored. At second order, $O(\varepsilon^2)$,

$$(2.15) \quad 2y_m^{(2)} - y_{m+1}^{(2)} - y_{m-1}^{(2)} = \frac{d^2 y_m^{(0)}}{d\xi^2} + (\Omega_0^2 y_m^{(2)} + \Omega_2^2 y_m^{(0)}) \begin{cases} M & m \neq 0, \\ M_0 & m = 0. \end{cases}$$

To proceed, we Fourier transform to get

$$(2.16) \quad \begin{aligned} & \tilde{y}^{(2)}(\alpha, \xi) - \frac{(M - M_0)\Omega_0^2 y_0^{(2)}}{[2\cos\alpha - 2 + \Omega_0^2 M]} \\ &= \frac{-\Omega_2^2 \tilde{Y}(\alpha) M f + (M - M_0)\Omega_2^2 f Y_0 - \tilde{Y}(\alpha) f_{\xi\xi}}{[2\cos\alpha - 2 + \Omega_0^2 M]}, \end{aligned}$$

where the tilde denotes a Fourier transform and $\tilde{y}^{(2)}(\alpha, \xi) = f(\xi)\tilde{Y}^{(2)}(\alpha)$ and $f_{\xi\xi} = d^2 f/d\xi^2$. Solvability, also called the Fredholm alternative or compatibility, as conventionally used in homogenization [28], then requires the right-hand side to be zero; inverting the Fourier transforms then gives the long-scale ordinary differential equation (ODE) for $f(\xi)$ as

$$(2.17) \quad 2(1 - \Omega_0^2 M/2) \frac{d^2 f}{d\xi^2} - 2M\Omega_2^2 f = 0.$$

This second-order ODE encodes the microscale behavior of the line defect valid on the continuum macroscale. Therefore this has homogenized the discrete problem into a simple continuum equation. Using this equation, we obtain an expression for Ω_2 and f , which gives the envelope of the amplitude variation along the lattice. This can be done in several ways; we shall illustrate this for Bloch waves in section 2.1, where f satisfies quasi-periodic boundary conditions and Ω_2 is found from a local dispersion relation. Alternatively, in section 2.2 a forcing, or equivalently a mass

variation, introduces an additional term in (2.17) leading to a differential eigenvalue problem for Ω_2 .

The asymptotics presented thus far are based upon the assumption (2.7) of periodicity on the microscale. It is also possible to develop another set of asymptotics if, on the microscale, we assume that the motion is locally out-of-phase, from one column of masses to the neighboring column; then $y_m(\xi, 0) = -y_m(\xi, \pm 1)$ and one obtains

$$(2.18) \quad 2(-3 + \Omega_\pi^2 M/2) \frac{d^2 f}{d\xi^2} + \Omega_\pi^2 [2M(3 - \Omega_\pi^2 M/2) + \Omega_\pi^2 (M - M_0)^2] f = 0,$$

where now, for clarity, we use Ω_π as the notation for the standing wave frequency at wavenumber $\kappa = \pi$, where κ is the Bloch wavenumber that is used in section 2.1, equation (2.20): Ω_π^2 is found from the relation

$$(2.19) \quad [(M\Omega_\pi^2/2 - 3)^2 - 1]^{1/2} = \frac{\Omega_\pi^2 (M - M_0)}{2}.$$

We now illustrate how (2.17) and (2.19) can be used.

2.1. Bloch waves. For the special situation of Bloch waves with quasi-periodic boundary conditions,

$$(2.20) \quad y_{n+1,m} = \exp(i\kappa) y_{n,m}$$

with Bloch wavenumber κ and $0 \leq \kappa \leq \pi$, one can return to the unapproximated difference equations (2.5) and deduce an exact dispersion relation relating frequency to Bloch wavenumber as

$$(2.21) \quad ([\cos \kappa - 2 + M\Omega^2/2]^2 - 1)^{1/2} = \Omega^2 (M - M_0)/2.$$

Rayleigh–Bloch waves require κ real, and we consider the branch with $\Omega \geq 2$ and $M_0 < M$; it is notable that if $M = M_0$,

$$(2.22) \quad \Omega^2 = \frac{2}{M} (3 - \cos \kappa),$$

which is the upper dashed branch along DA of the dispersion relations of the perfect lattice; in the notation of Figure 2(d) this is the lowest curve of Figure 3(a) folded back upon itself. As an aside, one can also use the asymptotic theory for the other branch along DA , but we do not show those results here. For clarity we have also marked AD on Figure 3(a), with A' as the periodic extension of A to π . It is notable that when the wavenumber $\kappa = 0$ there are standing waves that are in-phase from one mass to the next, and when $\kappa = \pi$ standing waves occur with masses out-of-phase from one to the next.

In this example, the line defect mode is born from this propagating branch of the perfect lattice, but due to the defect it then has spatial exponential decay in a direction perpendicular to the line defect. For two different values of M_0 , the masses of the line defect, this decay is shown in Figures 3(b,c); as M_0 departs further from unity the localization of the Rayleigh–Bloch wave to the line is enhanced. The frequency of the wave is entirely within the stop-band for $M_0 < M(1 - 1/\sqrt{2})$, and otherwise is partly embedded within the propagating branches of the perfect lattice. The consequences of this are shown in section 4, where we show that energy can leak from the surface wave to the bulk and vice versa. Figures 3(b,c) also show that the

FIG. 3. (a) Dispersion curves for the Rayleigh-Bloch wave guided along the line defect with $M = 1$ and M_0 taking the values $0.25, 1 - 1/\sqrt{2}, 0.5, 0.75, 0.9, 1$ with the arrow showing the direction of increasing M_0 . Solid lines are taken from the exact dispersion relation (2.21) and the dashed lines from the asymptotics in (2.23) and (2.24). Panels (b) and (c) show the displacements of two modes in the m direction, i.e., perpendicular to the defect.

asymptotic eigenfunctions clearly oscillate along the defective line as the amplitude decreases when moving away from the defect line ($m = 0$).

To relate this to the asymptotics, one notes that the Bloch condition (2.20) in the two-scale notation requires $f(\xi + \varepsilon) = \exp(i\kappa)f(\xi)$ for which $f(\xi) = \exp(i\xi)$ is a solution if ε is identified with κ . From the asymptotics one then finds, after some algebra, that the surface wave frequency, from (2.17), is locally

$$(2.23) \quad \Omega^2 = \Omega_0^2 - \kappa^2 \frac{(1 - M\Omega_0^2/2)}{M}$$

for $\kappa \ll 1$ and locally, from (2.18),

$$(2.24) \quad \Omega^2 = \Omega_\pi^2 + 2(\pi - \kappa)^2 \frac{(-3 + M\Omega_\pi^2/2)}{2M(3 - M\Omega_\pi^2/2) + \Omega_\pi^2(M - M_0)^2}$$

as $\kappa \rightarrow \pi$; here Ω_π is the frequency for waves perfectly out-of-phase on the microscale. These asymptotic results are shown as the dashed curves in Figure 3(a). The lowest solid curve in Figure 3(a) is the dashed line in Figure 2(d); as noted earlier, a different periodicity causes that curve to be folded back upon itself. Amusingly, one can operate

FIG. 4. A defective line defect embedded within a bulk lattice of identical masses is depicted, which is the situation treated in section 2.2. The defective mass has mass M_* , the line defect in which it is placed has masses M_0 , and the bulk lattice has masses M .

using two neighboring columns as the microscale cell and then have a double eigenvalue at $\kappa = \pi/2$ as a new standing wave frequency; the resulting asymptotics are then linear (see [11] for the repeated roots theory in a different context) and can be developed in the neighborhood of $\kappa = \pi/2$.

2.2. A defective defect. In the related area of photonics it is often the case [18] that defects are introduced as lines of points within a structure to either guide or focus light. We now consider the discrete version of coalescing point and line defects.

To demonstrate the versatility of the asymptotics, we now consider a line defect that is itself defective (as shown in Figure 4); that is, the line $m = 0$ again consists of masses M_0 embedded within a lattice of masses M , but the line itself has a single mass changed at, say, $n = 0$, so that it is M_* there. Intriguingly, this defective line defect will also have localized defect states with an eigenfrequency outside that of the surface wave spectrum. These defect states have exponential decay, away from the mass M_* , along the line defect. We now find these asymptotically.

The governing equation (2.5) is modified to
(2.25)

$$y_{n+1,m} + y_{n-1,m} + y_{n,m+1} + y_{n,m-1} - 4y_{n,m} = -\Omega^2 y_{n,m} \begin{cases} M & m \neq 0, \\ M_0(1 + \varepsilon\alpha\delta_{n,0}) & m = 0, \end{cases}$$

with $\delta_{n,0}$ as the Kronecker delta.

The following two cases naturally arise and illustrate the technique.

(i) If $M > M_0 > M_*$: we define $\varepsilon = 1 - M_*/M_0$ where $\alpha = -1$, so here the mass of the single defect is less than that of the line defect and surrounding lattice. After following through with the two-scale procedure, using $\delta_{n,0} = \varepsilon\delta(\xi)$, and assuming locally out-of-phase behavior ($y_m(\xi, 0) = -y_m(\xi, \pm 1)$), the ODE that governs the motion along the line defect is, from (2.25),

$$(2.26) \quad 2(-3 + \Omega_\pi^2 M/2) \frac{d^2 f}{d\xi^2} + \Omega_2^2 [2M(3 - \Omega_\pi^2 M/2) + \Omega_\pi^2 (M - M_0)^2] f - \Omega_\pi^4 (M - M_0) M_0 \alpha \delta(\xi) f = 0,$$

where Ω_π is found by solving $(\frac{M}{2}\Omega_\pi^2 - 3) - 1 = \frac{\Omega_\pi^4}{4}(M - M_0)^2$. Equation (2.26) has

FIG. 5. A defective defect state for (a) $M = 1, M_0 = 0.25, M_* = 0.2, \varepsilon = 0.2$ for which numerical simulations give $\Omega = 5.092178$ and the asymptotics $\Omega \sim 5.087701$, and (b) $M = 1, M_0 = 0.25, M_* = 0.3, \varepsilon = 0.2$ for which numerical simulations give $\Omega = 2.996830$ and the asymptotics give $\Omega \sim 2.995146$. In panels (a) and (b), the solid lines show the localized modes derived by the numerics, while the dashed lines represent the envelope, f , obtained by the asymptotics. The local oscillations on the microscale are out-of-phase and in-phase in (a) and (b), respectively.

an exact solution in the form $f = \exp(-\beta|\xi|)$ [20], where the decay rate β is given by

$$(2.27) \quad 4\beta(-3 + \Omega_\pi^2 M/2) = \Omega_\pi^4 (M - M_0)M_0$$

and

$$(2.28) \quad 2(-3 + \Omega_\pi^2 M/2)\beta^2 = -\Omega_2^2 [2M(3 - \Omega_\pi^2 M/2) + \Omega_\pi^2 (M - M_0)^2].$$

The eigenfrequency is thus

$$(2.29) \quad \Omega = \sqrt{\Omega_\pi^2 + \varepsilon^2 \Omega_2^2} + O(\varepsilon^4),$$

and comparisons of this solution versus numerical solutions for $M = 1, M_0 = 0.25, M_* = 0.2$ are shown in Figure 5(a). The numerical solutions are found by turning the infinite number of discrete difference equations (2.5) into a standard matrix eigenvalue problem, truncated to a large finite size, typically 100×100 , with convergence checked by increasing this value.

The displacements of the localized mode found numerically show locally antiperiodic behavior as expected. The dashed line in Figure 5(a) shows $f(\xi)$ which clearly captures the long-scale envelope of the numerically observed decay. This localized defect state occurs at frequency $\Omega = 5.092178$, which lies above the dispersion curve shown in Figure 3(a) and is in the nonpropagating region for both the main lattice and the perfect line defect, suggesting that a nonembedded trapped mode exists in this case. One can actually use the theory of positive operators, as in [16], to prove that this occurs for the ODE of (2.26).

(ii) If $M > M_* > M_0$: we define $\varepsilon = M_*/M_0 - 1$ and set $\alpha = +1$; thus the introduced mass is heavier than those of the line defect but lighter than those of the main lattice. After again following the two-scale procedure, we have

$$(2.30) \quad 2(1 - \Omega_0^2 M/2)f_{\xi\xi} - f[2M\Omega_2^2 + \Omega_0^4 (M - M_0)M_0\delta(\xi)] = 0.$$

FIG. 6. Dispersion curves for the wave guided along the defective line defect are depicted when $M = 1, M_0 = 0.25, 0 \leq M_* < 0.25$ (case (i) of the main text) and $\varepsilon = 1 - \frac{M_*}{M_0}$ in (a) and when $M = 1, M_0 = 0.25, 0.25 < M_* \leq 0.5$ (case (ii) of the main text) and $\varepsilon = \frac{M_*}{M_0} - 1$ in (b). The solid lines represent the solution obtained from the asymptotics, while the crosses show the numerical solution. Ω_U and Ω_L represent frequencies lying in the upper stop-band and in the lower propagating region, respectively, as ε increases.

In this case the local microscale behavior is in-phase as one moves from one column of masses to its neighbors; there will again be a localized mode, this time at a frequency below the surface wave mode and so potentially at a frequency for which waves propagate within the main lattice. The exponential solution $f = \exp(-\beta|\xi|)$ applies again and the decay is found from

$$(2.31) \quad -4\beta(1 - \Omega_0^2 M/2) = \Omega_0^4(M - M_0)M_0$$

with

$$(2.32) \quad (1 - \Omega_0^2 M/2)\beta^2 = M\Omega_0^2.$$

This localized defect state is shown in Figure 5(b) illustrating the excellent agreement between the numerical and asymptotic solutions even for the relatively large value of ε : $\varepsilon = 0.2$.

In Figure 6 the asymptotic frequency values for the localized mode occurring at frequencies Ω_U above, and Ω_L below, those of the surface wave frequencies are shown as crosses. For comparative purposes, full numerical simulations were performed and the localized states extracted; the comparison is excellent for ε values ($\varepsilon < 0.5$): as ε increases further the asymptotics move away from the numerical solution and therefore fail, as indeed is expected for an asymptotic theory based upon $\varepsilon \ll 1$. Figure 6(a) shows the trapped mode (Ω_U) existing in the nonpropagating regime; looking at the surface wave dispersion curve in Figure 3(a), this is closest to values of that curve at the $\kappa = \pi$ edge of the Brillouin zone where out-of-phase oscillatory behavior is exhibited, and hence this out-of-phase behavior in the localized defect state is expected. The frequency Ω_L , presented in Figure 6(b), is another trapped mode, this time potentially embedded in the propagating region of the main lattice, and near the other edge ($\kappa = 0$) of the Brillouin zone; thus we expect it to behave locally in-phase. These conclusions from the asymptotics are born out by full numerical simulations.

3. Asymptotics perpendicular to the line defect. We now change tack and attack the line defect problem from a different angle; here we now use two scales but in the m direction perpendicular to the line defect. Since the surface wave dispersion curves shown in Figure 3(a) are a modification of the dashed line along AD shown for the perfect lattice of Figure 2(d), it is natural to consider perturbing away from that solution. This will be less versatile than our previous asymptotic approach: in section 2 the asymptotic technique employed used the standing wave frequencies Ω_0 and Ω_π , at an edge point ($\kappa = 0$ or $\kappa = \pi$) of the reciprocal Brillouin zone, and required only these frequencies and the corresponding eigensolutions. Those asymptotics could then be used to consider more general settings such as localized defect states. In contrast, the asymptotic method used in this section requires the frequency at every wavenumber κ individually along the reciprocal Brillouin zone for the perfect lattice, i.e., without the line defect; we derive only one asymptotic expansion for frequency Ω . The crucial difference between the two techniques is the amount of information they require, as the former just assumes conditions at the edges, while the latter requires conditions imposed throughout the range of values of the wavenumber.

In order to proceed with an asymptotic approach, we again introduce two scales characterizing the lattice. The long scale, which we will now denote by η , is conveniently set using the relative mass variation between the grid and the defect: we set $\varepsilon = \frac{1}{\hat{\alpha}}(1 - \frac{M}{M_0})$, where $\hat{\alpha}$ is a constant and $\varepsilon \ll 1$, and $\eta = \varepsilon m$ to be continuous. The short scale m_1 is given by the elementary cell structure involving the four displacements $y_{n,m+m_1}$; $m_1 = -1, 0, 1, 2$. These two scales are now assumed to be independent of each other and allow us to set

$$(3.1) \quad \begin{aligned} y_{n,m+m_1} &= y_n(\eta + m_1\varepsilon, m_1) \\ &\sim y_n(\eta, m_1) + m_1\varepsilon y_{n_\eta}(\eta, m_1) + \frac{1}{2}(m_1\varepsilon)^2 y_{n_{\eta\eta}}(\eta, m_1) + \cdots \end{aligned}$$

By assuming that the motion, on the microscale of the elementary cell, is that of locally standing waves oscillating across the cell, we impose the following antiperiodic constraints for out-of-phase standing waves in the vertical (m) direction: $y_n(\eta, \pm 1) = -y_n(\eta, 0)$.

The mass variation in the asymptotic problem is treated by changing the right-hand side of (2.5) to

$$\begin{aligned} &-M\Omega^2 y_n(\eta, 0) + (M - M_0)\Omega^2 \varepsilon \delta(\eta) y_n(\eta, 0) \\ &= -M\Omega^2 y_n(\eta, 0) - M\Omega^2 (1 + \hat{\alpha}\varepsilon^2 \delta(\eta)) y_n(\eta, 0), \end{aligned}$$

which motivates the earlier definition of $\varepsilon = \frac{1}{\hat{\alpha}}(1 - \frac{M}{M_0})$.

After adjusting (2.5) in the two-scale notation and carrying out the appropriate substitutions for the Bloch and antiperiodic conditions, we obtain

$$(3.2) \quad (2 \cos \kappa - 6) y_n(\eta, 0) - \varepsilon^2 y_{n_{\eta\eta}}(\eta, 0) = -M\Omega^2 (1 + \hat{\alpha}\varepsilon^2 \delta(\eta)) y_n(\eta, 0);$$

Bloch conditions have been assumed such that $y_{n+1,m} = e^{i\kappa} y_{n,m}$. The next step is to use the ansatz

$$(3.3) \quad y_n(\eta) \sim y_n^{(0)}(\eta) + \varepsilon y_n^{(1)}(\eta) + \varepsilon^2 y_n^{(2)}(\eta) + \cdots \quad \text{and} \quad \Omega^2 \sim \Omega_0^2 + \varepsilon \Omega_1^2 + \varepsilon^2 \Omega_2^2 + \cdots,$$

which leads to a hierarchy of equations in powers of ε . These new differential equations are solved order-by-order in ε .

The leading-order problem is just

$$(3.4) \quad (2 \cos \kappa - 6)y_n^{(0)} = -M\Omega_0^2 y_n^{(0)},$$

from which the frequency of standing waves

$$(3.5) \quad \Omega_0^2 = \frac{6 - \cos \kappa}{M}$$

emerges directly (cf. (2.22)), and $y_n^{(0)}(\eta) = f(\eta)Y_n^{(0)}$, with $f(\eta)$ to be determined.

By separating the terms of different scales in the first-order equation, we are led to having $y_n^{(1)} = f_1(\eta)Y_n^{(1)}$ and

$$(3.6) \quad (2 \cos \kappa - 6)y_n^{(1)} + M\Omega_0^2 y_n^{(1)} = -M\Omega_1^2 y_n^{(0)}$$

for which solvability forces $\Omega_1^2 = 0$, and $f_1(\eta)$ is unknown and not required in the subsequent analysis.

The second-order problem after the separation of scales reads as

$$(3.7) \quad [(2 \cos \kappa - 6) + M\Omega_0^2]y_n^{(2)} = -M\Omega_2^2 y_n^{(0)} + y_{m\eta\eta}^{(0)} + M\Omega_0^2 \hat{\alpha} \delta(\eta) y_n^{(0)},$$

which is solvable only if the right-hand side equals zero. This condition leads to the differential eigenvalue problem

$$(3.8) \quad f_{\eta\eta} - M(\Omega_2^2 + \Omega_0^2 \hat{\alpha} \delta(\eta))f = 0,$$

where $f_{\eta\eta}$ denotes $\frac{d^2 f}{d\eta^2}$, relating f and Ω_2^2 . The solution of (3.8) is again $f(\eta) = -\exp(-\frac{M\Omega_0^2 \hat{\alpha}}{2}|\eta|)$ with $\Omega_2^2 = \frac{M\hat{\alpha}^2 \Omega_0^4}{4}$. Hence, the modified frequency is given asymptotically as

$$(3.9) \quad \Omega^2 \sim \frac{6 - 2 \cos \kappa}{M} + \varepsilon^2 \hat{\alpha}^2 \frac{(6 - 2 \cos \kappa)^2}{4M} + \dots$$

The numerical and asymptotic squared frequencies are plotted and compared in Figure 7(a). There we observe that as ε decreases, numerics and asymptotics have, as one would naturally expect, better agreement, while as ε increases the asymptotics move further away from the numerics.

In Figures 7(b,c) the same parameter values are chosen as in Figures 3(b,c), with the crucial difference being that the asymptotics of this section capture the exponential decay in the m direction away from the line defect, shown as the dashed curves in Figures 7(b,c).

4. Illustrative examples. We now consider some numerical examples to illustrate how the physical ideas of this paper can be used in interpretation. The numerical approach is to, again, create a matrix problem from the difference equations (2.5), but now with the line defects making the outline of a square. When truncating, one naturally has to be careful how the, artificially introduced, edges of the domain influence the solution, particularly with propagating solutions potentially reflecting off the edges. This issue is to overcome using a variant of perfectly matched layers (PML) [8] which are, for continua, the method of choice. For discrete systems, in one dimension, for a simple line of masses with forcing we take

$$(4.1) \quad y_{n+1} + y_{n-1} - 2y_n + M\Omega^2 \left(1 - \frac{\sigma(n)}{i\Omega}\right)^2 y_n = \delta_{n,0}$$

FIG. 7. (a) Dispersion curves for the wave guided along the line defect with $M = 1$, M_0 taking the values 0.25, 0.5, 0.9, and ε is 0.75, 0.5, 0.1, respectively. Solid lines are taken from the exact dispersion relation (2.21) and the crosses from the asymptotics in (3.9). Panels (b) and (c) show the displacements of two modes in the m direction, i.e., perpendicular to the defect.

FIG. 8. A bulk lattice of identical masses with line defects outlining a square is depicted as used in section 4. The size of the numerical grid is, however, substantially larger with 603×603 masses and the inner square being 200×200 masses.

on a lattice $-N \leq n \leq N$ with $\sigma(n) = 0$ for $|n| < N_{pml}$, so the PML is active only toward the edge of the truncated domain. In PML computations for continuous systems it is often observed, and indeed proved [34], that nonlinear dependence in σ is advantageous. Here we take $\sigma(n) = (N_{pml} - n)^q/N$ for $n > N_{pml}$ and a symmetric formula for $n < -N_{pml}$; in computations, $q = 2$ and $N_{pml} = 60$. The efficacy of this approach is checked versus exact solutions in one dimension, and its extension to two dimensions is similarly verified.

We consider a two-dimensional lattice of equal masses ($M = 1$) with masses of M_0 set along the edges of a square, as shown in Figure 8. We excite this system of masses by vibrating the mass at the lower left corner of the square with different frequencies. Once again applying PML in a similar manner to the one-dimensional problem, we obtain the difference equations

$$(4.2) \quad \begin{aligned} & y_{n+1,m} + y_{n-1,m} + y_{n,m+1} + y_{n,m-1} - 4y_{n,m} \\ & = \delta_{n,-201} - \Omega^2 \left(1 - \frac{\sigma(n)}{i\Omega} \right)^2 \begin{cases} M_0 & \text{for square,} \\ M & \text{otherwise,} \end{cases} \end{aligned}$$

where $-N \leq n, m \leq N$ with $\sigma(n) = 0$ for $|n|, |m| < N_{pml}$, so as shown in Figures 9 and 10, the PML becomes active outside the white rectangles. The defective masses M_0 form a square when $-201 \leq n \leq 201$ and $-201 \leq m \leq 201$. We proceed to solve them via matrix inversion.

With $M_0 = 0.25$ we can use Figure 2(d) to see that waves propagate in the main lattice for $0 < \Omega < 2.83$; there is then a stop-band as the frequency increases further. The insertion of the line defect introduces a surface wave dispersion curve, as shown in Figure 3(a), with propagation for frequencies $3.02 < \Omega < 4.95$. We choose frequencies $\Omega = 1.98, 2.5, 3$, and 4 , and from the dispersion curves anticipate that for particular frequencies defect modes will exist.

In Figure 9(a) the lattice is excited at $\Omega = 1.98$; this is very close to a flat band in the dispersion diagram, which is of independent interest (see [4]), and standing waves form entirely along the diagonals, centered at the forcing point, of the bulk lattice. We observe that energy travels along the nondefective masses outside and inside the box, while the inner square also allows for this to happen. The line defects around the edges have minimal effect apart from creating an image source at the opposite corner from the forcing.

Another frequency within the propagating regime, $\Omega = 2.5$, was examined, resulting in Figure 9(b). At this frequency we notice that energy propagates outward from the source, as expected, and some energy is transmitted within the square of defective masses. Again the line defects are rather passive and mainly act simply to reflect waves from the inner square.

Figures 9(c,d) depict the scenarios for two frequencies lying in the stop-band of the perfect bulk lattice, i.e., with the line defects absent. In the first case where $\Omega = 3$ (see Figure 9(c)), there is some energy at the vibration source which exponentially decays within a small vicinity around the source, as is predicted for frequencies in the nonpropagating regime. At this frequency the surface waves do not propagate either.

However, for $\Omega = 4$ we are in a stop-band for the bulk lattice, but within the propagating region of the line defects. In Figure 9(d), there is no energy traveling along the lattice except along the square of defective masses, showing that this is an excellent approach for guiding wave energy.

Now we consider the same problem with the same setting, but with $M_0 = 0.5$. As before, we look at Figure 3(a) and observe that surface waves are predicted at frequencies $2.3 < \Omega < 3.56$. It is essential to note that part of the surface wave dispersion curve now lies in the propagation region; hence the frequencies for which surface waves exist are not entirely in the stop-band as before when $M_0 = 0.25$. In other words, we expect to have nonembedded trapped modes in the overlapping set of values of Ω , $2.3 < \Omega < 2.83$. Again, numerical simulations are generated for 2.5 and 3 ; the other two frequencies are of less interest and are omitted for brevity.

FIG. 9. The absolute value of the displacements $|y_n|$ for an infinite lattice, constructed by identical masses $M = 1$ and embedded defective masses $M_0 = 0.25$ set in a square, is excited at different frequencies $\Omega = 1.98, 2.5, 3, 4$. The source is placed at the lower left-hand corner of the square. The PML method was used to avoid reflection or other edge effects: the white lines show the position at which the PML scheme becomes active.

FIG. 10. The absolute value of the displacements $|y_n|$ for an infinite lattice, constructed by identical masses $M = 1$ and embedded defective masses $M_0 = 0.5$ set in a square, is excited at different frequencies $\Omega = 2.5, 3$. The source is placed at the lower left-hand corner of the square. The PML method was used to generate these figures and avoid reflection or other edge effects. The white lines show the position at which the PML scheme becomes active.

In Figure 10(a), where $\Omega = 2.5$ is within the propagating regime of the bulk lattice, but also corresponds to a surface wave frequency, traveling waves appear throughout the nondefective masses in the lattice. Along the defective masses of the square, some energy is captured and remains trapped in standing waves traveling around the edge of the square, which creates image sources at each corner; this is clearly not an efficient frequency at which to transport energy. For $\Omega = 3$, which is in the stop-band, for the bulk but not for surface waves, we again have wave energy trapped to propagate along the line defects; this is due to the same reason as in Figure 9(d).

5. Conclusions. The asymptotic procedure, outlined by [12], has been extended to investigate Rayleigh–Bloch waves along line defects introduced into discrete two-dimensional lattices. High-frequency homogenization is based upon a two-scale approach that captures the microstructural information of a periodic inhomogeneous medium used in wave propagation problems and finds asymptotic homogenized continuum equations valid on the macroscale. This mathematical technique takes advantage of the scale separation by “transferring” the knowledge of the multiple scaling encoded by standing waves in the microscale to the macroscale. The short scale representing the microstructure properties has to be significantly smaller than the macroscale. The resulting high-frequency homogenization models are capable of encapsulating localization effects, such as recognizing localized defect modes.

In the surface wave problems presented here, embedded and nonembedded trapped modes, in the sense of being outside or within the frequency range for which waves propagate within a perfect lattice, were successfully detected by employing discrete high-frequency homogenization. This is verified by comparing asymptotic results to numerical solution of the full system. Frequencies associated with localized defect modes are found in the high-frequency regime. It is notable that we have concentrated on one branch of Rayleigh–Bloch waves; there is another branch of solutions that have a dispersion relation passing through the origin, and the asymptotic theory works equally well for them.

Two different approaches were demonstrated for tackling the same surface wave problem in sections 2.1 and 3. The first was based on perturbing about the edge points ($\kappa = 0$ and $\kappa = \pi$) of the reciprocal Brillouin zone, leading to two asymptotic expansions of frequency (2.17), (2.18). The other was developed by perturbing for each wavenumber κ individually along the reciprocal Brillouin zone, obtaining a single asymptotic expansion for Ω (3.8). The former approach requires information only about the edge points, making it more applicable and effective when considering more general settings, while the latter imposes conditions throughout the range of values of the wavenumber. As a result the former captures and produces a very good asymptotic solution near the edges and is less effective for the values in the middle of the reciprocal Brillouin zone. The latter, even though more demanding, can be more efficient in certain cases, as it can replicate the exact frequency behavior throughout the whole range of wavenumbers.

There are many variations or extensions to the surface wave problems examined here. In particular, time dependence could be introduced to the problems, allowing us to find trapped modes linked to time-dependent problems. Surface wave problems similar to those of the main text could be investigated involving different types of defects in the square lattice, such as point defects or defective blocks of masses. To this extent, varying the location of the defect, i.e., setting the line defect at the edge of a semi-infinite lattice, may also affect the resulting homogenized differential equation in the macroscale. High-frequency homogenization used here, for localization in two-

dimensional structures of a specific geometry, also allows for generalizations to three dimensions or to other engineering structures such as frames [22, 25, 29], relying on understanding the behavior of an elementary lattice cell. The two-scale approach is not limited to the square lattice geometry, and therefore other lattice geometries constructed by triangles or hexagons [22, 35] could also be solved using this formalism.

The asymptotic theory used here for treating discrete problems with the occurrence of localization phenomena can be applied to analogous continuous problems. Continuum models of structured interfaces can be considered and solved based on the same concept. For example, the equivalent corresponding case of the discrete lattice with a defective line could be replaced by an infinite array of parallel-plates (i.e., finite lines) with a plate defect, i.e., a line missing or being semi-infinite or a linear array of circular holes with one missing. Such extensions are currently underway.

Acknowledgments. The authors gratefully acknowledge helpful conversations with Julius Kaplunov, Mehul Makwana, and Evgeniya Nolde that aided in the development of this work.

REFERENCES

- [1] J. D. ACHENBACH, *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam, 1984.
- [2] T. ANTONAKAKIS AND R. V. CRASTER, *High-frequency asymptotics for microstructured thin elastic plates and platonics*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 468 (2012), pp. 1408–1427.
- [3] H. A. ATWATER AND A. POLMAN, *Plasmonics for improved photovoltaic devices*, Nature Mater., 9 (2010), pp. 205–213.
- [4] M. V. AYZENBERG-STEPANENKO AND L. I. SLEPYAN, *Resonant-frequency primitive waveforms and star waves in lattices*, J. Sound Vibration, 313 (2008), pp. 812–821.
- [5] N. BAKHVALOV AND G. PANASENKO, *Homogenization: Averaging Processes in Periodic Media*, Kluwer, Amsterdam, 1989.
- [6] C. P. J. BARTY, M. KEY, J. BRITTEN, R. BEACH, G. BEER, C. BROWN, S. BRYAN, J. CAIRD, T. CARLSON, J. CRANE, J. DAWSON, A. C. ERLANDSON, D. FITTINGHOFF, M. HERMANN, C. HOAGLAN, A. IYER, L. JONES, II, I. JOVANOVIĆ, A. KOMASHKO, O. LANDEN, Z. LIAO, W. MOLANDER, S. MITCHELL, E. MOSES, N. NIELSEN, H.-H. NGUYEN, J. NISSEN, S. PAYNE, D. PENNINGTON, L. RISINGER, M. RUSHFORD, K. SKULINA, M. SPAETH, B. STUART, G. TIETBOHL, AND B. WATTELLIERET, *An overview of LLNL high-energy short-pulse technology for advanced radiography of laser fusion experiments*, Nuclear Fusion, 44 (2004), pp. S266–S275.
- [7] A. BENSOUSSAN, J. L. LIONS, AND G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.
- [8] J. P. BERENGER, *A perfectly matched layer for the absorption of electromagnetic waves*, J. Comput. Phys., 114 (1994), pp. 185–200.
- [9] L. BRILLOUIN, *Wave Propagation in Periodic Structures: Electric Filters and Crystal Lattices*, 2nd ed., Dover, New York, 1953.
- [10] R. V. CRASTER, J. KAPLUNOV, E. NOLDE, AND S. GUENNEAU, *High frequency homogenization for checkerboard structures: Defect modes, ultra-refraction and all-angle-negative refraction*, J. Opt. Soc. Amer. A, 28 (2011), pp. 1032–1041.
- [11] R. V. CRASTER, J. KAPLUNOV, AND A. V. PICHUGIN, *High-frequency homogenization for periodic media*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 466 (2010), pp. 2341–2362.
- [12] R. V. CRASTER, J. KAPLUNOV, AND J. POSTNOVA, *High-frequency asymptotics, homogenisation and localisation for lattices*, Quart. J. Mech. Appl. Math., 63 (2010), pp. 497–519.
- [13] J. A. DESANTO, *Scattering from a periodic corrugated structure II. Thin comb with hard boundaries*, J. Math. Phys., 13 (1972), pp. 336–341.
- [14] D. V. EVANS AND C. M. LINTON, *Edge waves along periodic coastlines*, Quart. J. Mech. Appl. Math., 46 (1993), pp. 643–656.
- [15] D. V. EVANS AND R. PORTER, *Penetration of flexural waves through a periodically constrained thin elastic plate in vacuo and floating on water*, J. Engrg. Math., 58 (2007), pp. 317–337.
- [16] D. GRIDIN, R. V. CRASTER, AND A. T. I. ADAMOU, *Trapped modes in curved elastic plates*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), pp. 1181–1197.

- [17] R. A. HURD, *The propagation of an electromagnetic wave along an infinite corrugated surface*, Canad. J. Phys., 32 (1954), pp. 727–734.
- [18] J. D. JOANNOPOULOS, S. G. JOHNSON, J. N. WINN, AND R. D. MEADE, *Photonic Crystals: Molding the Flow of Light*, 2nd ed., Princeton University Press, Princeton, NJ, 2008.
- [19] M. D. KELZENBERG, S. W. BOETTCHER, J. A. PETYKIEWICZ, D. B. TURNER-EVANS, M. C. PUTNAM, E. L. WARREN, J. M. SPURGEON, R. M. BRIGGS, N. S. LEWIS, AND H. A. ATWATER, *Enhanced absorption and carrier collection in Si wire arrays for photovoltaic applications*, Nature Mater., 9 (2010), pp. 239–244.
- [20] M. MAKWANA AND R. V. CRASTER, *Localised point defect states in asymptotic models of discrete lattices*, Quart. J. Mech. Appl. Math., 66 (2013), pp. 289–316.
- [21] A. A. MARADUDIN, ED., *Structured Surfaces as Optical Metamaterials*, Cambridge University Press, Cambridge, UK, 2011.
- [22] P. G. MARTINSSON AND A. B. MOVCHAN, *Vibrations of lattice structures and phononic band gaps*, Quart. J. Mech. Appl. Math., 56 (2003), pp. 45–64.
- [23] C. C. MEI, J.-L. AURIAULT, AND C.-O. NG, *Some applications of the homogenization theory*, Adv. Appl. Mech., 32 (1996), pp. 278–345.
- [24] E. W. MONTROLL AND R. B. POTTS, *Effects of defects on lattice vibrations*, Phys. Rev. (2), 100 (1955), pp. 525–543.
- [25] E. NOLDE, R. V. CRASTER, AND J. KAPLUNOV, *High frequency homogenization for structural mechanics*, J. Mech. Phys. Solids, 59 (2011), pp. 651–671.
- [26] R. A. PALA, J. WHITE, E. BARNARD, J. LIU, AND M. L. BRONGERSMA, *Design of plasmonic thin-film solar cells with broadband absorption enhancements*, Advanced Materials, 21 (2009), pp. 3504–3509.
- [27] G. PANASENKO, *Multi-scale Modelling for Structures and Composites*, Springer, Dordrecht, 2005.
- [28] G. PAVLIOTIS AND A. M. STUART, *Multiscale Methods: Averaging and Homogenization*, Springer, New York, 2008.
- [29] A. S. PHANI, J. WOODHOUSE, AND N. A. FLECK, *Wave propagation in two-dimensional periodic lattices*, J. Acoust. Soc. Amer., 119 (2006), pp. 1995–2005.
- [30] R. PORTER AND D. V. EVANS, *Rayleigh-Bloch surface waves along periodic gratings and their connection with trapped modes in waveguides*, J. Fluid Mech., 386 (1999), pp. 233–258.
- [31] R. PORTER AND D. V. EVANS, *Embedded Rayleigh-Bloch surface waves along periodic rectangular arrays*, Wave Motion, 43 (2005), pp. 29–50.
- [32] K. SAKODA, T. UETA, AND K. OHTAKA, *Numerical analysis of eigenmodes localized at line defects in photonic lattices*, Phys. Rev. B, 56 (1997), pp. 14905–14908.
- [33] E. SANCHEZ-PALENCIA, *Non-homogeneous Media and Vibration Theory*, Springer-Verlag, Berlin, 1980.
- [34] E. A. SKELTON, S. D. M. ADAMS, AND R. V. CRASTER, *Guided elastic waves and perfectly matched layers*, Wave Motion, 44 (2007), pp. 573–592.
- [35] A. SPARAVIGNA, *Phonons in conventional and auxetic honeycomb lattices*, Phys. Rev. B, 76 (2007), 134302.