where α'_{K-1} is obtained as the derivative with respect to γ , evaluated as $\gamma \rightarrow 0$, of (28). As can be easily verified, the above is nothing but the small risk limit (i.e., as $\gamma \rightarrow 0$) of (29).

General Risk-Aversion Case ($\gamma > 0$): The following result helps bring to light a manifestation of aversion to risk in the DM or controller; its proof is given in the Appendix.

Lemma 4.3: Let $\gamma > 0$, then for all $k > 1$

$$\ln(\alpha_k) > \gamma \cdot \alpha'_k. \quad (33)$$

□

Notice that the decision to replace a unit involves an uncertain, and therefore a risky, investment in that the unit being replaced may actually be in working condition, or it may subsequently fail. This is reflected in (29), (32), and (33) in that a risk-neutral DM or controller may decide to replace a unit for values of R higher than a risk-averse DM or controller would.

APPENDIX

Proof of Lemma 4.3: From the recursion (28) it follows that

$$\alpha_k = \sum_{l=0}^{k-2} (1-\theta)^l \theta e^{(k-1-l)\gamma C} + (1-\theta)^{k-1} \quad (34a)$$

$$\alpha'_k = \sum_{l=0}^{k-2} (1-\theta)^l \theta (k-1-l)C. \quad (34b)$$

Now, it also follows from (28) that $\alpha_k|_{\gamma=0} = 1$, and therefore (34a) is a convex combination of exponentials (the last term corresponding to $e^{0 \cdot \gamma C}$). Therefore, since $\ln(\cdot)$ is a strictly concave function, we have that

$$\begin{aligned} \ln(\alpha_k) &= \ln \left(\sum_{l=0}^{k-2} (1-\theta)^l \theta e^{(k-1-l)\gamma C} + (1-\theta)^{k-1} \right) \\ &> \sum_{l=0}^{k-2} (1-\theta)^l \theta \ln(e^{(k-1-l)\gamma C}) = \gamma \alpha'_k. \end{aligned} \quad (35)$$

□

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A General Minimal Residual Krylov Subspace Method for Large-Scale Model Reduction

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Abstract—This paper considers approximating a given n th-order stable transfer matrix $G(s)$ by an r th-order stable transfer matrix $G_r(s)$ in which $n \gg r$, and where n is large. The Arnoldi process is used to generate a basis to a part of the controllability subspace associated with the realization of $G(s)$, and a residual error is defined for any approximation in this subspace. We establish that minimizing the $\mathcal{L}_\infty$ norm of this residual error over the set of stable approximations leads to a 2-block distance problem. Finally, the solution of this distance problem is used to construct reduced-order approximate models. The behavior of the algorithms is illustrated with a simple example.

Index Terms—Distance problem, GMRES, large-scale systems, model reduction.

I. INTRODUCTION

Consider a linear time-invariant stable (LTIS) state-space model of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \quad (1)$$

where $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times p}$, $C \in \mathcal{R}^{q \times n}$, and in which $x(t)$ is the state vector, and $u(t)$ and $y(t)$ represent the input and output,

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respectively. It is assumed that the eigenvalues of A have negative real parts. The model reduction problem considered in this paper finds an approximate LTIS state-space model

$$\dot{x}_r(t) = A_r x_r(t) + B_r u(t), \quad y_r(t) = C_r x_r(t)$$

where $A_r \in \mathcal{R}^{r \times r}$, $B_r \in \mathcal{R}^{r \times p}$, $C_r \in \mathcal{R}^{q \times r}$, and in which $x_r(t)$ is of dimension $r \ll n$ and the eigenvalues of A_r have negative real parts. Well-established model reduction methods, such as optimal Hankel norm [7] and balanced truncation [11], require the solution of the Lyapunov equations $AP + PA' + BB' = 0$ and $A'Q + QA + C'C = 0$. When n is large, say $n > 200$, these methods are intractable because of the prohibitive storage requirements and a computational burden of $O(n^3)$ for each equation. Furthermore, these techniques do not in general exploit the sparseness of A . If A can be cheaply transformed to a diagonal form, so that P and Q can be easily calculated, and if storage of PQ is not a major problem, then the block balancing techniques of [18] may be used to calculate reduced-order models.

In order to overcome some of these problems, this paper considers the use of Krylov subspace techniques for the solution of the model reduction problem. These techniques are a class of iterative methods originally proposed for the solution of large eigenvalue problems. They have been successfully applied to the solution of approximation problems in various fields of scientific research. Good surveys of research activity in this area are [2] and [13]. The motivation for using these techniques is to enable low-order approximate models to be efficiently calculated while carrying out the computations in the low dimension. The application of Krylov subspace methods to the computation of minimal realizations for linear time-invariant (LTI) systems has been considered in [3] and [12]. More recently, [4] and [16] suggested the use of these techniques for large-scale model reduction; however, no algorithms were given. Krylov subspace model reduction algorithms were presented in [1] and [10]. The present work extends these results by addressing two new issues. First, an element of optimality is introduced by considering the general minimal residual (GMRES) problem, and second, the problem of ensuring the stability of the reduced-order model is resolved.

Section II gives a summary of the notation used. Section III gives a brief review of Krylov subspace methods. The residual error incurred is also defined. Section IV presents an algorithm for minimizing the $\mathcal{L}_\infty$ norm of this residual error and shows that this GMRES problem is equivalent to a 2-block distance problem. A cancellation analysis demonstrates that the solution to this distance problem gives stable low-dimensional approximations. Section V gives an example and the conclusions appear in Section VI.

II. NOTATION

The notation is mostly standard and is summarized here for convenience.

$\mathcal{R}, \mathcal{C}, \mathcal{R}(s)$ are real, complex numbers, field of proper, real rational functions of s . $\mathcal{F}^{p \times m}$ represents the set of $p \times m$ matrices with elements in $\mathcal{F}$. $\lambda_i(A)$, A' show the eigenvalue of square matrix A , complex conjugate transpose of $A \in \mathcal{C}^{p \times m}$. $A > 0$ denotes A is positive definite. $\mathcal{L}_\infty, \|\cdot\|$ are the space of functions bounded on the $j\omega$ axis, norm of matrices in $\mathcal{L}_\infty$. $\mathcal{H}_\infty^+(\mathcal{H}_\infty^-)$ is the subspace of $\mathcal{L}_\infty$ and functions analytic and bounded in the open right (left) half-plane.

Transfer matrices will be represented by uppercase boldface letters and with the dependence on s suppressed. Associated with every $G(s) \in \mathcal{R}(s)^{q \times p}$ is a state-space realization $\mathbf{G} = D + C(sI - A)^{-1}B$, where $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times p}$, $C \in \mathcal{R}^{q \times n}$, and $D \in \mathcal{R}^{q \times p}$,

and where $n \geq$ McMillan degree of $\mathbf{G}$. The alternative notation

$$\mathbf{G} \stackrel{s}{=} (A, B, C, D) \quad \text{or} \quad \mathbf{G} \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

is also used. In this notation, the para-Hermitian conjugate of $\mathbf{G}$ is given by $G^\sim(s) := G(-\bar{s})' \stackrel{s}{=} (-A', -C', B', D')$. Occasionally we write

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (2)$$

in which $R_{ij} \stackrel{s}{=} (A, B_j, C_i, D_{ij})$ for $i, j = 1, 2$. If $\mathbf{G}^{-1} = \mathbf{G}^\sim$, then $\mathbf{G}$ is allpass and satisfies $\mathbf{G}\mathbf{G}^\sim = \mathbf{G}^\sim\mathbf{G} = I$. Occasionally, we say that $\mathbf{G}$ is γ -allpass for some positive γ . This is taken to mean that $\gamma^{-1}\mathbf{G}$ is allpass. The γ -ball of $\mathcal{H}_\infty^+$ (matrices $\mathbf{G} \in \mathcal{H}_\infty^+$ for which $\|\mathbf{G}\|_\infty \leq \gamma$) is denoted by $\gamma^{-1}\mathcal{B}\mathcal{H}_\infty^+$. (Whenever there is no possibility of confusion, we drop matrix dimensions.) A basis change in a state-space realization of $\mathbf{G} \stackrel{s}{=} (A, B, C, D)$ means the similarity transformation $\mathbf{G} \stackrel{s}{=} (S^{-1}AS, S^{-1}B, CS, D)$. If $\mathbf{R}$ is partitioned as in (2) and $\mathbf{U}$ has appropriate dimensions, we define the lower linear fractional map as $\mathcal{F}(\mathbf{R}, \mathbf{U}) := \mathbf{R}_{11} + \mathbf{R}_{12}\mathbf{U}(\mathbf{I} - \mathbf{R}_{22}\mathbf{U})^{-1}\mathbf{R}_{21}$, provided that $[\mathbf{I} - \mathbf{R}_{22}(\infty)\mathbf{U}(\infty)]$ is invertible. If $\mathcal{U}$ is a set, then $\mathcal{F}(\mathbf{R}, \mathcal{U})$ denotes the set $\{\mathcal{F}(\mathbf{R}, \mathbf{U}) : \mathbf{U} \in \mathcal{U}\}$.

III. KRYLOV SUBSPACE TECHNIQUES

This section establishes the type of low-order approximations used and characterizes the residual error associated with any low-order approximation.

By taking Laplace transforms, assuming zero initial conditions, the transfer matrix associated with the model (1) is given by $\mathbf{G} = C(sI - A)^{-1}B = C\mathbf{G}_B$, where $\mathbf{G}_B = (sI - A)^{-1}B$. $\mathbf{G}_B$ can be considered as the solution of the linear system of equations

$$(sI - A)\mathbf{G}_B = B. \quad (3)$$

The focus of this paper is to find low-order approximate solutions to (3). If $\mathbf{G}_{B,m}$ is such a solution, we will consider $\mathbf{G}_m := C\mathbf{G}_{B,m}$ to be an approximation to $\mathbf{G}$ since $\mathbf{G} = C\mathbf{G}_B$. This approximation process is reminiscent of the one exploited for solving linear systems of equations via Krylov subspace techniques [14]. Following [10], this paper considers approximate solutions to (3) of the form

$$\mathbf{G}_{B,m} = V_m \mathbf{F}_m \quad (4)$$

where $m \ll n$ and the columns of V_m form an orthonormal basis for the Krylov subspace

$$\mathcal{K}_m(A, B) = \text{span} \{[B \quad AB \quad A^2B \quad \cdots \quad A^{m-1}B]\}.$$

$\mathbf{F}_m$ is a stable transfer matrix to be determined below. The remainder of this section is devoted to the efficient calculation of V_m and the criteria used to select $\mathbf{F}_m$. The well-established Arnoldi algorithm [8] is used to compute an orthonormal basis V_m for the Krylov subspace $\mathcal{K}_m(A, B)$. The basic outline of the Arnoldi process is given in terms of $A \in \mathcal{R}^{n \times n}$ and $B \in \mathcal{R}^{n \times p}$.

Arnoldi Process

- Compute $Q_1 R_1 = B$ and set $p_1 :=$ number of columns of Q_1 (QR factorization).
- Do $j = 1, \dots, m$
 - a) Set $V_j = [Q_1 \ Q_2 \ \dots \ Q_j]$.
 - b) Compute $\begin{bmatrix} A_{1j} \\ \vdots \\ A_{jj} \end{bmatrix} = V_j' A Q_j$.
 - c) $Q_{j+1} A_{j+1,j} = A Q_j - \sum_{k=1}^j Q_k A_{kj}$ (QR factorization), and $p_{j+1} :=$ number of columns of Q_{j+1} .
- End Do.

The QR factorizations of the Arnoldi process can be computed via the modified Gram–Schmidt (MGS) orthogonalization process. In the event that B does not have full column rank, MGS does not generally yield an orthogonal basis for B . The QR factorization is then given by $B\Pi = [\hat{Q}_1 \ \hat{Q}_2] \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ 0 & 0 \end{bmatrix}$ in which $\hat{R}_{11}$ is upper triangular and Π is a permutation matrix. In such a situation, we set $Q_1 := \hat{Q}_1$ and $R_1 := [\hat{R}_{11} \ \hat{R}_{12}]\Pi'$. Finally, $p_1 := \text{rank}(Q_1) = \text{rank}(B)$, and hence V_1 has only p_1 columns. Similar remarks apply to the QR factorization of $AQ_j - \sum_{k=1}^j Q_k A_{kj}$ in the main loop of the Arnoldi process. Observe that the upper-triangular structure of $\hat{R}_{j1}$ is lost since Π' permutes the columns of $[\hat{R}_{j1} \ \hat{R}_{j2}]$. The matrices R_{j+1} are nonsquare and will have row rank of p_{j+1} . We refer to any rank drop in B or $AQ_j - \sum_{k=1}^j Q_k A_{kj}$ as a *curable* breakdown since it does not adversely affect the Arnoldi process if tackled in the manner just described. For the most part, we will assume that this type of breakdown does not occur, consequently m steps of the Arnoldi process generate $V_m \in \mathcal{R}^{n \times mp}$. For further implementation details of the block Arnoldi process, the reader is referred to [9].

By construction, the Arnoldi process yields an orthonormal basis $V_m = [Q_1 \ Q_2 \ \dots \ Q_m]$ for the Krylov subspace $K_m(A, B)$. Defining the $mp \times mp$ block upper-Hessenberg matrix $A_m = (A_{ij})_{j=1:m}^{i=1:m}$, where $A_{ij} = 0$ for $i > j + 1$, it is easy to verify that

$$\begin{aligned} AV_m &= V_m A_m + Q_{m+1} A_{m+1,m} E_m' \\ B &= V_m B_m, \quad A_m = V_m' A V_m \end{aligned} \quad (5)$$

where E_m is a matrix of the last p columns of the mp identity matrix and $B_m := [R_1' \ 0_{p \times l}]'$, where $l = p(m-1)$. Assuming that the approximate solution to (3) has the form $\mathbf{G}_{B,m} := V_m \mathbf{F}_m$ for some arbitrary function $\mathbf{F}_m$, then the residual error function associated with this solution is defined as

$$\begin{aligned} \mathbf{R}_m(\mathbf{F}_m) &:= B - (sI - A)\mathbf{G}_{B,m} = B - (sI - A)V_m \mathbf{F}_m \\ &= [V_m \ Q_{m+1}] \begin{bmatrix} B_m - (sI - A_m)\mathbf{F}_m \\ A_{m+1,m} E_m' \mathbf{F}_m \end{bmatrix} \end{aligned} \quad (6)$$

on using (5). Krylov-based solution techniques select $\mathbf{F}_m$ so as to give $\mathbf{R}_m(\mathbf{F}_m)$ desirable properties. For example, the Galerkin-type conditions imposed in [10] sought to determine low-order approximate solutions $\mathbf{G}_{B,m} = V_m \mathbf{F}_m$ such that $\mathbf{R}_m(\mathbf{F}_m)$ is orthogonal to the Krylov subspace $K_m(A', C')$. If the columns of the orthonormal matrix W_m span $K_m(A', C')$, and assuming that $W_m' V_m$ is nonsingular, then [10] showed that the Galerkin-type low-order approximate model is given by

$$\mathbf{G}_m \stackrel{s}{=} \left[\begin{array}{c|c} A_m + (W_m' V_m)^{-1} W_m' Q_{m+1} A_{m+1,m} E_m' & B_m \\ \hline C V_m & 0 \end{array} \right]. \quad (7)$$

A drawback of this procedure is that there is no guarantee that the low-order approximate model $\mathbf{G}_m$ is stable. In contrast, the next section presents an alternative criterion that guarantees a stable

low-order approximate model while simultaneously incorporating an element of optimality.

IV. A GMRES MODEL REDUCTION ALGORITHM

This section determines a low-order LTIS approximation of the form given in (4) such that $\|\mathbf{R}_m(\mathbf{F}_m)\|_\infty$ is minimized. More specifically, we address the following GMRES problem.

Problem 4.1 (GMRES): Find the minimal residual norm

$$\gamma_{\text{opt}} = \inf_{\substack{\mathbf{F}_m \in \mathcal{H}_\infty^+ \\ \mathbf{F}_m(\infty) = 0}} \|\mathbf{R}_m(\mathbf{F}_m)\|_\infty \quad (8)$$

and $\forall \gamma \geq \gamma_{\text{opt}}$, find the set of approximations $\mathcal{F}_{m,\gamma} := \{\mathbf{F}_m \in \mathcal{H}_\infty^+ : \mathbf{F}_m(\infty) = 0, \|\mathbf{R}_m(\mathbf{F}_m)\|_\infty \leq \gamma\}$.

Remark 4.1: The need for a stable and strictly proper $\mathbf{F}_m$ follows from the constraint that $\mathbf{G}_m$ must be stable and strictly proper since $\mathbf{G}$ is stable and strictly proper. If $\mathbf{F}_m$ is stable and strictly proper, $(sI - A_m)\mathbf{F}_m$ is stable and proper; consequently, the $\mathcal{L}_\infty$ norm in (8) is well defined.

Since the columns of $[V_m \ Q_{m+1}]$ are orthonormal, (6) gives

$$\begin{aligned} \|\mathbf{R}_m(\mathbf{F}_m)\|_\infty &= \left\| \begin{bmatrix} B_m - (sI - A_m)\mathbf{F}_m \\ A_{m+1,m} E_m' \mathbf{F}_m \end{bmatrix} \right\|_\infty \\ &= \|T_{11} - T_2 \mathbf{F}_m\|_\infty \end{aligned} \quad (9)$$

where

$$T_{11} := \begin{bmatrix} B_m \\ 0 \end{bmatrix}, \quad T_2 := \begin{bmatrix} sI - A_m \\ -A_{m+1,m} E_m' \end{bmatrix}.$$

Thus the GMRES problem has the form of a model matching problem, which is equivalent to a distance problem [5]. The form is nonstandard since T_2 is improper and $\mathbf{F}_m$ must be strictly proper. However, $T_2 \mathbf{F}_m$ is proper and stable whenever $\mathbf{F}_m$ is strictly proper and stable. Based on this observation, the following results demonstrate that Problem 4.1 is equivalent to a 2-block distance problem.

Lemma 4.1: Suppose that m steps of the Arnoldi process have been taken, and let $C_m := A_{m+1,m} E_m'$. Then (A_m, C_m) is detectable. Furthermore, if no breakdown has occurred, then (A_m, C_m) is observable.

Proof: The result follows from the stability of A , the staircase form of $[A_m^T \ C_m^T]^T$, and [15]. $\square$

Lemma 4.2: Suppose that m steps of the Arnoldi process have been taken and that no breakdown has occurred. Let $X_m = X_m' > 0$ be the unique stabilizing solution to the algebraic Riccati equation

$$A_m' X_m + X_m A_m + E_m A_{m+1,m}' A_{m+1,m} E_m' - X_m^2 = 0 \quad (10)$$

and define

$$\begin{aligned} \mathbf{R} &= \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A_R & B_{R1} \\ \hline C_{R1} & 0 \\ C_{R2} & 0 \end{array} \right] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} -(A_m - X_m)' & X_m B_m \\ \hline A_{m+1,m} E_m' X_m^{-1} & 0 \\ I & 0 \end{array} \right]. \end{aligned} \quad (11)$$

Then $\mathbf{R} \in \mathcal{H}_\infty^-$, and the minimum residual norm is given by

$$\gamma_{\text{opt}} = \inf_{\mathbf{Q} \in \mathcal{H}_\infty^+} \left\| \begin{bmatrix} R_{11} \\ R_{21} + \mathbf{Q} \end{bmatrix} \right\|_\infty. \quad (12)$$

Furthermore, if $\gamma \geq \gamma_{\text{opt}}$ and the set of all 2-block extensions of $\mathbf{R}$ is defined as

$$\mathcal{Q}_\gamma := \left\{ \mathbf{Q} \in \mathcal{H}_\infty^+ : \left\| \begin{bmatrix} R_{11} \\ R_{21} + \mathbf{Q} \end{bmatrix} \right\|_\infty \leq \gamma \right\} \quad (13)$$

then, with $\mathbf{T} := sI - A_m + X_m$, the set of all GMRES approximations, $\mathcal{F}_{m,\gamma}$, is given by

$$\mathcal{F}_{m,\gamma} = \{\mathbf{F}_m = \mathbf{T}^{-1}(B_m - \mathbf{Q}) : \mathbf{Q} \in \mathcal{Q}_\gamma\}. \quad (14)$$

Proof: Since the pair (A_m, I_m) is controllable and the pair $(A_m, A_{m,m+1}E'_m)$ is observable by Lemma 4.1, there exists a symmetric positive definite solution X_m to (10) such that $\text{Re } \lambda_i(A_m - X_m) < 0, \forall i$ [17]. Thus $\mathbf{R} \in \mathcal{H}_\infty^-$ follows from (11). We change nothing by rewriting (9) as

$$\|\mathbf{R}_m(\mathbf{F}_m)\|_\infty = \|T_{11} - T_{12}\mathbf{T}\mathbf{F}_m\|_\infty \quad (15)$$

where $T_{12} = T_2\mathbf{T}^{-1}$. A simple manipulation using (10) gives

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{T} &= (-sI - A'_m)(sI - A_m) + E_m A'_{m+1,m} \\ &\quad \cdot A_{m+1,m}E'_m = \mathbf{T}_2^{-1}\mathbf{T}_2. \end{aligned}$$

Hence, $\mathbf{T}_2^{-1}\mathbf{T}_2 = I$. It then follows from [7, Th. 5.2] that there exists an allpass completion $\mathbf{T}_\perp$ such that $\mathbf{T}_0 := [\mathbf{T}_\perp \quad \mathbf{T}_{12}]$ is allpass. Using (10) it can be verified that

$$\begin{aligned} \mathbf{T}_0 &= [\mathbf{T}_\perp \quad \mathbf{T}_{12}] \\ &\stackrel{s}{=} \left[\begin{array}{c|c} A_m - X_m & X_m^{-1}E_m A'_{m+1,m} \\ \hline -X_m & 0 \\ -A_{m+1,m}E'_m & I \end{array} \right] I. \end{aligned} \quad (16)$$

Hence, it follows from (15) that

$$\begin{aligned} \|\mathbf{R}_m(\mathbf{F}_m)\|_\infty &= \left\| T_{11} - \mathbf{T}_0 \begin{bmatrix} 0 \\ \mathbf{T}\mathbf{F}_m \end{bmatrix} \right\|_\infty \\ &= \left\| \mathbf{T}_0^{-1}T_{11} - \begin{bmatrix} 0 \\ \mathbf{T}\mathbf{F}_m \end{bmatrix} \right\|_\infty \end{aligned} \quad (17)$$

where the second equality follows from the fact that $\mathbf{T}_0$ is allpass and the $\mathcal{L}_\infty$ -norm preserving property of allpass matrices. A calculation using (16) verifies that

$$\mathbf{T}_0^{-1}T_{11} = \begin{bmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} + B_m \end{bmatrix}$$

and, together with (17), this implies that

$$\|\mathbf{R}_m(\mathbf{F}_m)\|_\infty = \left\| \begin{bmatrix} \mathbf{R}_{11} \\ \mathbf{R}_{21} + (B_m - \mathbf{T}\mathbf{F}_m) \end{bmatrix} \right\|_\infty.$$

This establishes that the $\mathcal{L}_\infty$ norm of the residual error in (6) may be recast as the $\mathcal{L}_\infty$ norm of a 2-block like structure. To establish (12) and (14) one needs to prove that

$$\{\mathbf{Q} := B_m - \mathbf{T}\mathbf{F}_m : \mathbf{F}_m \in \mathcal{H}_\infty^+, \mathbf{F}_m(\infty) = 0\} \equiv \mathcal{H}_\infty^+. \quad (18)$$

Suppose $\mathbf{F}_m \in \mathcal{H}_\infty^+$ and $\mathbf{F}_m(\infty) = 0$ and define $\mathbf{Q} = B_m - \mathbf{T}\mathbf{F}_m$. Since $\mathbf{T} = sI - A_m + X_m$, it follows that $\mathbf{Q}$ is proper and hence is in $\mathcal{H}_\infty^+$. Conversely, suppose that $\mathbf{Q} \in \mathcal{H}_\infty^+$ and define $\mathbf{F}_m := \mathbf{T}^{-1}(B_m - \mathbf{Q})$ so that $\mathbf{Q} = B_m - \mathbf{T}\mathbf{F}_m$. Then since $\mathbf{T}^{-1}(\infty) = 0$ and $\mathbf{T}^{-1} \in \mathcal{H}_\infty^+$, it follows that $\mathbf{F}_m \in \mathcal{H}_\infty^+$ and $\mathbf{F}_m(\infty) = 0$. This proves (18) and establishes (12) and (14).

Lemma 4.2 demonstrates that the solution of the GMRES problem is equivalent to finding (12) and the set of stable extensions (13). The solution of the distance problem can be found in [6]. γ_{opt} can be calculated to any desired degree of accuracy using γ -iteration. It is outside the scope of this paper to develop the solution to the 2-block distance problem; however, to effect further analysis in the present setting, we need the following theorem which summarizes the properties of the solution.

Theorem 4.3: Let $\mathbf{R}$ be as given in (11) and assume that $\gamma \geq \gamma_{\text{opt}}$. Then we have the following.

- 1) There exists an embedding of $\mathbf{R}$ of the form

$$\mathbf{H} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & 0 \\ \mathbf{R}_{21} + \mathbf{Q}_{21} & \mathbf{R}_{22} + \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{bmatrix}$$

such that $\mathbf{H}$ is γ -allpass; $\mathbf{R}_{ij} \in \mathcal{H}_\infty^-, i, j = 1, 2$; and $\mathbf{Q}_{ij} \in \mathcal{H}_\infty^+, i = 2, 3, j = 1, 2, 3$. Furthermore, the set of all 2-block stable extensions of $\mathbf{R}$ is given by

$$\mathcal{Q}_\gamma = \mathcal{F}[\mathbf{Q}_a, \mathcal{U}] \quad (19)$$

where $\mathcal{U} := \gamma^{-1}\mathcal{BH}_\infty^+$, and $\mathbf{Q}_a = \begin{bmatrix} \mathbf{Q}_{31}^* & \mathbf{Q}_{33}^* \\ \mathbf{Q}_{21} & \mathbf{Q}_{23} \end{bmatrix}$.

- 2) There exist real matrices

$$\begin{aligned} A_H &= \begin{bmatrix} A_R & 0 \\ 0 & A_Q \end{bmatrix}, & P_H &= \begin{bmatrix} P_1 & P_3 \\ P'_3 & P_2 \end{bmatrix} \\ Q_H &= \begin{bmatrix} Q_1 & Q_3 \\ Q'_3 & Q_2 \end{bmatrix} \\ B_H &= \begin{bmatrix} B_{R1} & B_{R2} & 0 \\ B_{Q1} & B_{Q2} & B_{Q3} \end{bmatrix}, & C_H &= \begin{bmatrix} C_{R1} & 0 \\ C_{R2} & C_{Q2} \\ 0 & C_{Q3} \end{bmatrix} \\ D_H &= \begin{bmatrix} 0 & \gamma I & 0 \\ D_{21} & 0 & D_{23} \\ D_{31} & 0 & 0 \end{bmatrix} \end{aligned}$$

such that

$$A_Q \in \mathcal{R}^{r \times r}, \quad r \leq mp, \quad \text{Re } \lambda_i(A_Q) < 0, \quad \forall i \quad (20)$$

$$\begin{aligned} A'_H Q_H + Q_H A_H + C'_H C_H &= 0 \\ C'_H D_H + Q_H B_H &= 0 \\ A_H P_H + P_H A'_H + B_H B'_H &= 0 \\ D_H B'_H + C_H P_H &= 0 \\ D_H D'_H &= D'_H D_H = \gamma^2 I \\ P_H Q_H &= Q_H P_H = \gamma^2 I. \end{aligned} \quad (21)$$

Furthermore

$$\mathbf{H} \stackrel{s}{=} \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array} \right]$$

and

$$\mathbf{Q}_a = \begin{bmatrix} \mathbf{Q}_{21} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{33} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_Q & B_{Q1} & B_{Q3} \\ \hline C_{Q2} & D_{21} & D_{23} \\ C_{Q3} & D_{31} & 0 \end{array} \right]. \quad (22)$$

Proof: See [6]. $\square$

The next result uses Theorem 4.3 to give a parameterization of all solutions to Problem 4.1.

Theorem 4.4: Let all variables be as in Problem 4.1, Lemma 4.2, and Theorem 4.3, and let $\gamma \geq \gamma_{\text{opt}}$. Then the set of all stable and strictly proper $\mathbf{F}_m$ such that $\|\mathbf{R}_m(\mathbf{F}_m)\|_\infty \leq \gamma$ is given by

$$\mathcal{F}_{m,\gamma} = \mathcal{F}[\mathbf{F}_a, \mathcal{U}] \quad (23)$$

where

$$\begin{aligned} \mathbf{F}_a &= \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} := \begin{bmatrix} \mathbf{T}^{-1}(B_m - \mathbf{Q}_{21}) & -\mathbf{T}^{-1}\mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{33} \end{bmatrix} \\ &\stackrel{s}{=} \left[\begin{array}{c|cc} A_Q & B_{Q1} & B_{Q3} \\ \hline Q_3 & 0 & 0 \\ C_{Q3} & D_{31} & 0 \end{array} \right]. \end{aligned} \quad (24)$$

Proof: Equation (23) and the second equality in (24) follow by substituting (19) into (14). It remains to prove the state-space realization in (24). A simple state-space calculation using (11) and (22) verifies that

$$\mathbf{F}_a \stackrel{s}{=} \left[\begin{array}{cc|cc} -A'_R & -C_{Q_2} & B_m - D_{21} & -D_{23} \\ 0 & A_Q & B_{Q_1} & B_{Q_3} \\ \hline I & 0 & 0 & 0 \\ 0 & C_{Q_3} & D_{31} & 0 \end{array} \right]. \quad (25)$$

Applying the basis change $S = \begin{bmatrix} I & Q_3 \\ 0 & I \end{bmatrix}$ to the state-space of (25) yields (26), as shown at the bottom of the page.

Expanding the allpass equations (21), we have (27) and (28), also shown at the bottom of the page.

Rearranging (10), pre- and post-multiplying by X_m^{-1} , and using (11)

$$A'_R(-X_m^{-1}) + (-X_m^{-1})A_R + C'_{R_1}C_{R_1} + C'_{R_2}C_{R_2} = 0. \quad (29)$$

Finally, subtracting (29) from the (1,1) block of (27) gives $A'_R(Q_1 + X_m^{-1}) + (Q_1 + X_m^{-1})A_R = 0$. This implies that $Q_1 = -X_m^{-1}$, since all the eigenvalues of A_R have positive real parts [7, Th. 3.3]. Equation (11), the (1,2) block of (27), and the (1,1) and (1,3) blocks of (28) imply that

$$\begin{aligned} A'_R Q_3 + Q_3 A_Q + C_{Q_2} &= 0, & -B_m + Q_3 B_{Q_1} + D_{21} &= 0 \\ Q_3 B_{Q_3} + D_{23} &= 0. \end{aligned}$$

Removing the uncontrollable modes corresponding to the first row and column of (26) gives (24). $\square$

Remark 4.2: Theorem 4.4 demonstrates that the GMRES model reduction problem may have more than one solution since the solution of the corresponding 2-block distance problem is, in general, not unique. For the purposes of this work, one optimal solution may be chosen by observing that $\mathbf{U} = 0 \in \mathcal{U}$; consequently, (23) and (24) imply that a GMRES approximation is given by

$$\mathbf{F}_m = \mathcal{F}(\mathbf{F}_a, \mathbf{U}) = \mathbf{F}_{11} \stackrel{s}{=} (A_Q, B_{Q_1}, Q_3, 0). \quad (30)$$

Hence, a GMRES approximation is given by $\mathbf{G}_m = CV_m \mathbf{F}_{11} \stackrel{s}{=} (A_Q, B_{Q_1}, CV_m Q_3, 0)$. It follows from (20) that $\mathbf{G}_m$ is stable and has McMillan degree $\leq mp$. Since $\mathbf{G}_m$ has the same McMillan degree as $\mathbf{F}_m$, and the McMillan degree of $\mathbf{F}_m$ is less than or equal to the sum of the McMillan degrees of $\mathbf{F}_a$ and $\mathbf{U}$ from (30), it follows that any nondynamic $\mathbf{U} \in \mathcal{U}$ will generate a stable $\mathbf{G}_m$ of McMillan degree $\leq mp$.

The following procedure summarizes the GMRES model reduction algorithm proposed in this section. We start with $\mathbf{G} = C(sI - A)^{-1}B$ in which $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times p}$, and $C \in \mathcal{R}^{q \times n}$ are given.

GMRES Model Reduction Algorithm

- Start: Specify a tolerance $\epsilon > 0$, set an integer parameter m .
- Perform m steps of the Arnoldi process to compute $A_m, B_m, A_{m+1,m}, V_m, Q_{m+1}$, and E_m .
- Compute the stabilizing solution X_m to (10) and define the 2-block system $\mathbf{R}$ via (11).
- Evaluate γ_{opt} in (12) using the γ -iteration [6]. If $\gamma_{\text{opt}} > \epsilon$, increase m , and continue the Arnoldi process. Otherwise, compute Q_a in (22) [6], and form the optimal generator $\mathbf{F}_a$ using (24).
- Choose nondynamic $\mathbf{U} \in \mathcal{U}$, e.g., $\mathbf{U} = 0$, and form the GMRES solution $\mathbf{G}_m = CV_m \mathcal{F}(\mathbf{F}_a, \mathbf{U})$, e.g., $\mathbf{G}_m = CV_m \mathbf{F}_{11} = CV_m Q_3(sI - A_Q)^{-1}B_{Q_1}$.

V. NUMERICAL EXAMPLE

This section illustrates, with the help of an example, the behavior of the GMRES model reduction algorithm. The tests were performed on a Sparc-10 Sun workstation using Pro-MATLAB 4.0 which carries out operations to a unit round off of 2.22×10^{-16} .

The problem is set up with $A \in \mathcal{R}^{n \times n}$, where $n = 100$, and the top left-hand 2×2 block of A is set to $\begin{bmatrix} -1 & 100 \\ -100 & -1 \end{bmatrix}$, while the remaining nonzero elements of A are uniformly distributed in $[0, -1]$, and are all located on the leading diagonal. B and C' are n -dimensional column vectors. The first five elements of B and C are uniformly distributed in $[0, 1]$, while the remaining elements are uniformly distributed in $[0, 1/25]$. Table I shows the evolution of the $\mathcal{L}_\infty$ norm of the residual error (6) as m is increased. E1 is the residual error $\mathcal{L}_\infty$ norm for the Galerkin-type scheme defined by (7) and developed in [10], while E2 is the residual error $\mathcal{L}_\infty$ norm for the GMRES algorithm. Table I indicates that E1 does not necessarily decrease in magnitude as m increases, since Galerkin-type conditions do not guarantee a nonincreasing evolution of E1. By definition, E2 is smaller than E1 and is nonincreasing. This is confirmed by the table. Observe that GMRES gives a much lower residual error $\mathcal{L}_\infty$ norm for small m . This suggests that GMRES may be more suitable in a potential restart scheme. The table also shows the evolution of the $\mathcal{L}_\infty$ norm of the forward error $\mathbf{G} - \mathbf{G}_m$ as m is increased. E3 is the forward error $\mathcal{L}_\infty$ norm incurred when using the Galerkin model reduction algorithm (7), E4 is the forward error $\mathcal{L}_\infty$ norm using GMRES, while E5 is the forward error $\mathcal{L}_\infty$ norm using the balanced truncation model reduction algorithm [11]. Table I shows that the balanced truncation scheme enjoys a rapid forward error $\mathcal{L}_\infty$ norm decay for increasing m , while the convergence rate of both Krylov subspace-based algorithms is slower.

Ideally, one would like to obtain a reduced-order model using the superior balanced truncation technique. However, the large-scale nature of the problem imposes constraints on the complexity of the calculations allowed in any model reduction algorithm. As in other Krylov subspace methods, a potential improvement in the convergence rate of the GMRES model reduction scheme may lie

$$\mathbf{F}_a \stackrel{s}{=} \left[\begin{array}{cc|cc} -A'_R & -(A'_R Q_3 + Q_3 A_Q + C_{Q_2}) & B_m - D_{21} - Q_3 B_{Q_1} & -(D_{23} + Q_3 B_{Q_3}) \\ 0 & A_Q & B_{Q_1} & B_{Q_3} \\ \hline I & Q_3 & 0 & 0 \\ 0 & C_{Q_3} & D_{31} & 0 \end{array} \right] \quad (26)$$

$$\left[\begin{array}{cc} A'_R Q_1 + Q_1 A_R + C'_{R_1} C_{R_1} + C'_{R_2} C_{R_2} & A'_R Q_3 + Q_3 A_Q + C'_{R_2} C_{Q_2} \\ A'_Q Q'_3 + Q_3 A_R + C'_{Q_2} C_{R_2} & A'_Q Q_2 + Q_2 A_Q + C'_{Q_2} C_{Q_2} + C'_{Q_3} C_{Q_3} \end{array} \right] = 0 \quad (27)$$

$$\left[\begin{array}{cc} Q_1 B_{R_1} + Q_3 B_{Q_1} + C'_{R_2} D_{21} & Q_1 B_{R_2} + Q_3 B_{Q_2} + \gamma C'_{R_1} \\ Q_3 B_{R_1} + Q_2 B_{Q_1} + C'_{Q_2} D_{21} + C'_{Q_3} D_{31} & Q_3 B_{R_2} + Q_2 B_{Q_2} & Q_3 B_{Q_3} + C'_{R_2} D_{23} \\ & & Q_2 B_{Q_3} + C'_{Q_2} D_{23} \end{array} \right] = 0 \quad (28)$$

TABLE I

RESIDUAL ERROR $\mathcal{L}_\infty$ NORMS USING GALERKIN (E1) AND GMRES (E2) MODEL REDUCTION ALGORITHMS AND FORWARD ERROR $\mathcal{L}_\infty$ NORMS USING GALERKIN (E3), GMRES (E4), AND BALANCED TRUNCATION (E5) SCHEMES

m	1	2	3	4	5
E1	5.1091	5.6085	0.3308	0.1016	0.0349
E2	1.4295	1.1813	0.3053	0.0884	0.0296
E3	2.0903	2.0017	0.2322	0.0854	0.0683
E4	2.0010	1.9932	0.0853	0.0608	0.0736
E5	0.8549	0.4743	0.0968	0.0324	0.0090

in embedding the current technique in a restart scheme, the subject of current research by the author.

VI. CONCLUSIONS

This paper has presented and tested a Krylov-based model reduction algorithm suitable for computing low-order approximate models for large-scale stable systems. In contrast to previously developed Krylov-based schemes, the algorithm ensures the stability of the reduced-order model while enjoying a minimal residual error $\mathcal{L}_\infty$ norm.

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Disturbance Decoupling in a Class of Linear Systems

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Abstract—The disturbance decoupling problem by state feedback (DDP) is solved for a class of linear systems where the left-invertible systems are included. Also obtained is a complete characterization of the solution as well as the parameterization of all corresponding state feedback matrices. Such characterization enables one to conclude about the solvability of DDP with stability and/or pole placement. A numerical method is proposed for computation of the controllers in an analytical form, suitable to the treatment of complementary design specifications. The solution of the disturbance decoupling problem by measurement feedback (DDPM) is also discussed and, based on the results obtained for DDP, a procedure for design of a reduced-order compensator is proposed.

Index Terms—Disturbance decoupling, geometric approach, linear systems, state/output feedback.

I. INTRODUCTION

In the framework of the geometric approach for linear control systems, the disturbance decoupling problem by state feedback (DDP) ranks foremost. Its theoretical solution constitutes a perfect example of the application of some of the fundamental concepts of this approach, such as (A, B) -invariant and controllability subspaces. Textbooks [1] and [2] contain the complete problem definition, as well as conditions for its solvability and the characterization of the related state feedback matrices.

In many practical situations, full state information is not available, which induces the formulation of the disturbance decoupling problem by measurement feedback (DDPM). The complete problem definition, solvability conditions, as well as the characterization of the corresponding compensators can be found in [2]–[4].

In spite of the existence of this characterization, few works have been concerned with the derivation of analytical expressions

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