Optimal Active Control and Optimization of a Wave Energy Converter

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Abstract—This paper investigates optimal active control schemes applied to a point absorber wave energy converter within a receding horizon fashion. A variational formulation of the power maximization problem is adapted to solve the optimal control problem. The optimal control method is shown to be of a bang-bang type for a power take-off mechanism that incorporates both linear dampers and active control elements. We also consider a direct transcription of the optimal control problem as a general nonlinear program. A variation of the projected gradient optimization scheme is formulated and shown to be feasible and computationally inexpensive compared to a standard NLP solver. Since the system model is bilinear and the cost function is not convex quadratic, the resulting optimization problem is not a quadratic program. Results will be compared with an optimal command latching method to demonstrate the improvement in absorbed power. All time domain simulations are generated under irregular sea conditions.

Index Terms—Wave energy, Optimal control, Projected gradient method

I. INTRODUCTION

IN recent years there has been active research in advanced control methods for oscillating body wave energy converters (WECs). Since the early 1970s, it has been recognized that these devices have a narrow bandwidth and early research used mechanical impedance matching schemes to maximize the velocity and hence the absorbed power from sinusoidal (or regular) waves [1]. Simple frequency domain analysis was used to derive optimal amplitude and phase conditions on the velocity of the device with respect to a sinusoidal wave excitation force. Often called *reactive* control in the wave energy literature [1], this method's theoretical optimal resonance condition, of having the velocity in phase with the sinusoidal force, results in unrealistically large amplitudes and large two-way energy transfers between the body and the power take-off (PTO) mechanism. This method's shortcomings include its inability to handle physical constraints [1], [2] and its non-applicability to systems with a nonlinear PTO.

In [3], only the phase criteria is met through latching control — during its oscillation, the body is latched (i.e. prevented from moving) when its velocity vanishes and released at a favorable time. Approximating latching numerically by a very large linear damper that can be switched on and off, [4] computes an optimal sequence of latching/unlatching commands to maximize extracted energy in a simplified model. Since then, optimal latching control has been widely applied to single

DOFs devices [2], [5]–[7] as well as WECs with multiple DOF [6], [8], [9]. However, as is known for multiple-DOF systems, the effectiveness of latching diminishes for an array of devices interacting with each other; the phase condition loses meaning and optimal power absorption no more requires all bodies to have a velocity in phase with the excitation force [10]. This has motivated some research in the application of advanced optimal active control schemes to wave energy. In this article, we use the standard control engineering terminology *active control* to refer to control mechanisms that are not passive. Consider a dynamical system with input $u(t)$ and output $y(t)$, where $y(t) = h(t, u(t))$, $t \in [0, \infty)$, $u, y \in \mathbb{R}^p$. The system is called passive if $u(t)^T y(t) \geq 0$, $\forall t \in [0, \infty)$ [11, Ch. 6]. The WEC controller can be seen as a dynamical system whose input is the WEC velocity and whose output is the control force it exerts on the WEC. A passive controller is a passive dynamical system. Control mechanisms that inject external energy could then be considered active.

Another passive method considers varying the damping coefficient of the PTO continuously in time; in practice, this is done in a discrete or pseudo-continuous way and results in a complex PTO with a lot of components. Motivated by the need to alleviate this problem, the work in [12] has shown via simulations that an on-off strategy with an optimal command gives more energy and therefore does at least as well as its continuous counterpart. Since this on-off strategy is practically implemented using a simple by-pass valve, it is called *declutching or unlatching* control.

The works in [13], [14] consider the use of an active force within the framework of model predictive control (MPC) and so are of importance to the present article. Both papers consider only an active element for the PTO and depend on the reformulation of an energy maximization problem to discrete-time model problems. In [13], the radiation force is expressed as a linear function of the WEC velocity. Using the velocity as the optimization variable, the discretized optimization problem over a finite prediction horizon is shown to be a positive semidefinite quadratic program in the discrete velocity values – a convex problem. The emphasis in [14] is on discretizing the system using a triangle-hold such that the objective function can be approximated with one where the optimization parameters become changes in the control input at each sampling time; the method employed allows the approximation of the objective function by a semidefinite quadratic cost. Regularization terms are also added to impose penalties on the control and its derivative. However, as will be shown in the present article, the optimal control is of bang-bang type when no displacement or velocity constraints are imposed or when they are inactive. This would exploit

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the practical advantage that bang-bang controllers can be implemented with simple on-off machinery; a triangle-hold implementation would not make use of this advantage.

Another recent work [15], considers the synthesis of optimal causal active controllers using the statistical characterisation of ocean waves. There, the emphasis is on the causality of the controller — the need to know future wave forces is alleviated. The wave power spectral density is approximated by a linear time-invariant system driven by a white noise process and this is augmented to the buoy dynamic model. The objective to maximise the average extracted electrical energy results in a non-standard LQG optimal control problem, which is solved using standard tools.

In the present article, which is mainly based on our previous work in [16], we consider a general optimal active control problem for a heaving point absorber. It is general in the sense that it considers a PTO with a controlled damping element in addition to the active control force considered by [13], [14]. This problem results in a bilinear system dynamics and a cost function that is not convex quadratic—the resulting optimization problem is not a quadratic program unlike the ones in [13], [14]. In addition to simulations as in [12], we show the on-off nature of the optimal controller theoretically in this general PTO setting. Moreover, this formulation can be generalized in a straightforward manner to devices moving in more degrees of freedom and with various control elements. Actuation and physical constraints are also easily incorporated in this setting. We will also formulate and use a globally convergent and computationally cheap gradient projection scheme for computing the control commands. We employ a state-of-the-art interior-point optimization software within a direct collocation method to solve the resulting nonlinear program for comparison and validation.

In Section II, we will discuss the dynamics of a heaving buoy and its state space model derived for control. Section III presents a variational formulation of the optimal control problem and methods to solve it. Finally, in Section IV an example device is used to demonstrate the computational gains from using a projected gradient method. Control feasibility and the improvement that optimal active control delivers over optimal latching control is also presented. PTO parameter optimization and its dependence on the control scheme used is discussed.

II. SYSTEM DYNAMICS

In this article, we consider a semi-submerged cylindrical point absorber constrained to move in heave only; see Fig. 1. A rigid body interacting with an inviscid, incompressible and irrotational fluid flow is assumed. Considering the sea bottom as an inertial reference, the vertical displacement of the buoy from the equilibrium (in the absence of waves) is represented by $\zeta(t)$. Then, the buoy displacement with time t is given as

$$M\ddot{\zeta}(t) = f_c(t) + f_h(t) + f_r(t) + f_{exc}(t), \quad (1)$$

where M is the mass of the body and f_c represents the vertical control force exerted on the buoy. The net hydrostatic restoring force due to buoyancy and gravity is given by f_h and is proportional to the displacement

$$f_h(t) = -C_h\zeta, \quad (2)$$

where the hydrostatic stiffness $C_h := \rho g S$, with ρ being the density of water, g gravitational acceleration and S the cross-sectional area of the buoy. The heave excitation force f_{exc} is the force exerted on the stationary body at equilibrium due to the interaction with the oncoming waves. The radiation forces f_r describe the forces due to the movement of the body itself in the absence of incident waves; changes in the momentum of the surrounding fluid and the resulting radiated waves give rise to net forces on the body. Assuming a linear water-body interaction and using velocity potential theory these forces can be linearly related to the displacement, velocity and acceleration of the buoy in the frequency domain; see [1] and references therein for the derivation of frequency domain transfer functions relating the velocities with radiation and excitation forces for some floating geometries in water.

Fig. 1. A schematic of heaving buoy point absorber WEC.

A time-domain approach models the radiation force using

$$f_r(t) = -\mu_\infty \ddot{\zeta}(t) - \int_{-\infty}^t k_r(t-\tau) \dot{\zeta}(\tau) d\tau, \quad (3)$$

also referred to as the Cummins equation [17]. The so-called infinite-frequency added mass μ_∞ represents an instantaneous force response of the fluid after an impulsive movement of the buoy. The convolution integral represents forces due to the transient fluid motion or radiated waves caused by the motion of the buoy. The impulse-response of the radiation force $k_r(\cdot)$ can be computed using time-domain simulations via software like WAMIT and ACHIL3D [6]. The equation of motion (1) can now be re-written as:

$$(M + \mu_\infty) \ddot{\zeta}(t) + \int_{-\infty}^t k_r(t-\tau) \dot{\zeta}(\tau) d\tau + C_h \zeta(t) = f_{exc}(t) + f_c(t). \quad (4)$$

In a control algorithm, (4) would have to be solved at each time. However, this computation is more efficiently calculated with an approximate state-space model for the convolution integral [18]. Considering the velocity $\dot{\zeta}(t)$ as the input of a linear time-invariant continuous-time system of order m and the integral approximation $y_r(t)$ as the output, we have:

$$\begin{aligned} \dot{z}_r(t) &= A_r z_r(t) + B_r \dot{\zeta}(t), z_r(0) = 0, \\ y_r(t) &= C_r z_r(t) \approx \int_{-\infty}^t k_r(t-\tau) \dot{\zeta}(\tau) d\tau, \end{aligned} \quad (5)$$

where the state $z_r(t) \in \mathbb{R}^m$, $A_r \in \mathbb{R}^{m \times m}$, $B_r \in \mathbb{R}^{m \times 1}$ and $C_r \in \mathbb{R}^{1 \times m}$. As in [14], we call this the *radiation subsystem*.

With a radiation subsystem of order $m = n - 2$ identified

in (5), the WEC system dynamics can be re-written in state space form as:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{1}{M+\mu_\infty}[f_{exc}(t) + f_c(t) - C_r x_{3:n}(t) - C_h x_1(t)], \\ \dot{x}_{3:n}(t) &= A_r x_{3:n}(t) + B_r x_2(t),\end{aligned}\quad (6)$$

where the notation $x_{a:b}$ is to be interpreted as ‘elements a to b of the state vector x ’ and the new state $x := [x_1 \ x_2 \ \dots \ x_n]^T = [\zeta \ \dot{\zeta} \ z_r^T]^T \in \mathbb{R}^n$ with the appropriate initial conditions. See [19] and references therein for methods of system identification – we use the time-domain method implemented in the Matlab function `imp2ss` and discussed in this reference. Like the radiation force, the excitation force has an integral representation, $f_{exc}(t) = \int_{-\infty}^{\infty} k_{exc}(t - \tau) \eta(\tau) d\tau$, where $\eta(\cdot)$ is the wave elevation at the buoy and $k_{exc}(\cdot)$ the excitation force impulse response function. The non-causality of $k_{exc}(\cdot)$ and methods to “causalize” it are discussed in [18]. This problem can be rectified either by predicting the wave elevation at the buoy sometime into the future or by measuring the wave elevation some distance ahead of the buoy in the direction of the wave propagation. A state space approximation can then be computed the same way as for the radiation integral. In the following, we assume all states are known. The design of observers for the radiation and excitation forces from position, velocity and other sensor information is the topic of ongoing work.

III. OPTIMAL CONTROL PROBLEM

The aim here is to examine the optimal control problem to be used within a receding horizon framework. The underlying basis of this method is an iterative, finite-time optimization of the plant model [20]. At any sampling instant, the measured state values are used as initial conditions to calculate an optimal input function or sequence and the associated future state trajectory. Therefore, at the root are an optimal control algorithm to find the input sequence, and an ordinary differential equation (ODE) solver to calculate the state trajectory. Here, we investigate the optimal control problem for maximizing energy extracted from a generic WEC. As in all the discussed literature, we assume that the excitation force is known in the prediction horizon. In practice, this is not true and an estimate would be used.

A. The Optimal Control Problem

Let us consider the WEC of (6) again. Here we assume that power is taken off through a damping force proportional to the velocity (with the constant damping coefficient B_{pto} being controlled proportionally through the input command $u_2(t) \in [0, 1]$) and a bounded active forcing element; i.e. $f_c(t) = -B_{pto} u_2(t) x_2(t) + u_1(t) G$, where $u_1 \in [-1, 1]$ and $G > 0$ is a (large) constant with a unit of force (N). The equation of motion (6) can then be re-written as:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{1}{M+\mu_\infty}[f_{exc}(t) + G u_1(t) - B_{pto} u_2(t) x_2(t) - C_r x_{3:n}(t) - C_h x_1(t)], \\ \dot{x}_{3:n}(t) &= A_r x_{3:n}(t) + B_r x_2(t),\end{aligned}\quad (7)$$

where $C := (C_h + k)$, k is the stiffness of an external spring system attached to the buoy, $u_1(t) \in [-1, 1]$ and $u_2(t) \in [0, 1]$. This dynamics is that of a *bilinear system*, i.e. it is linear in the input and linear in the state, but not jointly linear in both. From here on, where convenient, we use the augmented input vector $u(t) := [u_1(t) \ u_2(t)]^T$ and the set $\mathbb{U} := \{u(\cdot) : u_1(t) \in [-1, 1] \text{ and } u_2(t) \in [0, 1], \forall t \in [t_0, t_f]\}$. The objective, at time $t = t_0$, is to maximize the energy E extracted over a future time interval $[t_0, t_f]$; we solve the optimal control problem:

$$\begin{aligned}OCP : \min_{u(\cdot) \in \mathbb{U}} \int_{t_0}^{t_f} \{-B_{pto} u_2(t) x_2^2(t) + G u_1(t) x_2(t)\} dt, \\ \text{subject to (7) and } x(t_0) = \hat{x} \text{ given.}\end{aligned}\quad (8)$$

The dynamic constraint of (7) can be added to the minimization problem using a Lagrange multiplier $\lambda \in \mathbb{R}^n$ as

$$J := -E + \int_{t_0}^{t_f} \lambda^T(t) [f(x(t), u(t), t) - \dot{x}(t)] dt \quad (9)$$

where $f(\cdot)$ is a vector representation of the right hand side of (7) and $x(t)$ the state. The Hamiltonian associated with the minimization of J then becomes [21, Sec. 2.3]:

$$\begin{aligned}H(x, u, \lambda, t) := & -B_{pto} u_2 x_2^2 + G u_1 x_2 + \lambda_1 x_2 + \frac{\lambda_2}{M + \mu_\infty} \{f_{exc}(t) + \\ & G u_1 - B_{pto} u_2 x_2 - C_r x_{3:n} - C_h x_1\} + \\ & \lambda_{3:n}^T (A_r x_{3:n} + B_r x_2).\end{aligned}\quad (10)$$

(t in u , x and λ is dropped for notational convenience).

Pontryagin’s minimum principle considers the above formulation and derives necessary (and sufficient) conditions for optimality based on the idea that small variations of a locally optimal control u should not decrease the objective function of the minimization problem. We consider an optimal input $u(\cdot) \in \mathbb{U}$ and an arbitrarily small admissible perturbation $\delta u(\cdot)$, i.e. $u(t) + \delta u(t) \in \mathbb{U}, \forall t \in [t_0, t_f]$ and $\|\delta u(t)\|_{L^1} < \varepsilon$, where for $v : [0, \infty) \rightarrow \mathbb{R}^m$, $\|v\|_{L^1} := \int_0^\infty \sum_{i=1}^m |v_i(t)| dt$ [22, Sec. 3.4]. The cost function can then be shown to satisfy:

$$\begin{aligned}J(u + \delta u) - J(u) = \int_{t_0}^{t_f} \{H(x, u + \delta u, \lambda; t) - H(x, u, \lambda; t)\} dt \\ + O(\varepsilon),\end{aligned}\quad (11)$$

where ε is a small number and the vector of adjoint variables λ satisfy the set of adjoint differential equations

$$\dot{\lambda}(t) = -\frac{\partial H^T}{\partial x}(x(t), u(t), \lambda(t), t), \quad (12)$$

and the final condition is $\lambda(t_f) = 0$, because the terminal cost is zero. Detailed derivations are available in [22, Ch. 3] and [21].

Generally, through the weak form of the PMP, a candidate (locally) optimal control law $u(\cdot)^*$ can be derived from the first order necessary condition $H_u := \partial H(\cdot)/\partial u = 0$ and sufficient conditions are verified using $\partial^2 H(\cdot)/\partial u^2$ or by substituting $u(\cdot)^*$ into the objective function. However, since both the performance measure in (8) and the dynamics (7) are linear in the control input, u does not appear in H_u . Therefore, it does not give us a candidate optimal control. A first order necessary condition for optimality, i.e. for $J(u + \delta u) - J(u)$ in (11) to be

non-negative, is then [22, Thm 3.4.2]

$$H(x(t), u(t)^*, \lambda(t); t) \leq H(x(t), u(t), \lambda(t); t), \quad \forall u(t) \in \mathbb{U}, \forall t \in [t_0, t_f], \quad (13)$$

where $H(\cdot)$ and $\lambda(\cdot)$ are as defined in (10), and (12), respectively. Simply put, the PMP states that the optimal control, and its corresponding state and co-state trajectories, must minimise the Hamiltonian for all time $t \in [t_0, t_f]$ and for all “neighbouring” admissible inputs.

In problems where the control is bounded, i.e. $\mathbb{U} := \{u(\cdot) : u(t) \in [u_{\min}, u_{\max}], \forall t \in [t_0, t_f]\}$, (13) allows us to show necessary conditions for optimality. Moreover, the system dynamics and the cost function being linear in the input makes the Hamiltonian affine in the control, i.e. it has the form:

$$H(x(t), u(t), \lambda(t); t) = l(x(t), \lambda(t), t) + \sigma(x(t), \lambda(t), t)^T u(t) \quad \forall t \in [t_0, t_f], \quad (14)$$

where $l(x(t), \lambda(t), t) \in \mathbb{R}$ and $\sigma(x(t), \lambda(t), t) \in \mathbb{R}^m$, $\forall x, \lambda, t$. The necessary condition of (13) then reduces to:

$$\sigma(x(t), \lambda(t), t)^T u^*(t) \leq \sigma(x(t), \lambda(t), t)^T u(t) \quad \forall u(t) \in \mathbb{U}, \forall t \in [t_0, t_f]. \quad (15)$$

This further simplifies to conditions on the components of the optimal input, namely:

$$u_i^*(t) = \begin{cases} u_{\min, i} & \text{if } \sigma_i(x(t), \lambda(t), t) > 0, \\ u_{\max, i} & \text{if } \sigma_i(x(t), \lambda(t), t) < 0, \\ \text{undetermined} & \text{if } \sigma_i(x(t), \lambda(t), t) = 0, \end{cases} \quad (16)$$

for $i = 1, \dots, m$, $\forall t \in [t_0, t_f]$.

It is clear that $\sigma(\cdot) = H_u(\cdot)$. The components $\sigma_i(\cdot)$ are called switching curves; the optimal input components switch from one boundary to the other at the zero crossings of the corresponding function. We say a singular arc occurs if any of the switching functions $\sigma_i(\cdot)$, $i = 1, \dots, m$, vanishes identically on an interval of nonzero measure in $[t_0, t_f]$. In such intervals, (16) does not determine the optimal input.

B. Analysis of Singular Arcs

Here, we show that the optimal control problem (8) with only input constraints has no singular arcs. Let us consider the switching functions for (10). Along a singular arc, one or more of the switching functions vanish and so the linear necessary conditions of the PMP (15) are not adequate to determine a unique optimal control candidate; infinitely many admissible control trajectories, $u(\cdot) \in \mathbb{U}$, trivially meet the conditions. In order to determine a control law along the arc, a high order maximum principle (HMP) has to be considered [21, Ch. 6], [23]. Suppose that the switching function $\sigma(x, \lambda; t) := H_u$ is identically zero over a finite interval $(t_1, t_2) \subset [t_0, t_f]$. Then all the time derivatives $\frac{d^q}{dt^q} H_u$, $q = 1, 2, \dots, \infty$, must also vanish over the same interval. By successive differentiation of the switching function, one may find the smallest integer q such that:

$$\begin{aligned} \frac{d^i}{dt^i} H_u &= 0, \quad \forall t \in [t_1, t_2], \quad i = 0, \dots, q \text{ and} \\ \frac{\partial}{\partial u} \left(\frac{d^q}{dt^q} H_u \right) &\neq 0, \text{ for some } t \in (t_1, t_2). \end{aligned} \quad (17)$$

If such a finite q exists, it must be even [21, Sec. 8.4] [23]. The variable p , with $q = 2p$ is called the order of singularity. Moreover, the candidate singular optimal control over $[t_1, t_2]$ and the corresponding $2n - q$ dimensional singular manifold of the (x, λ) -space are computed by substituting the state and adjoint dynamics (7) and (12), respectively, into (17). In addition, an admissible optimal control along a singular arc of order p must satisfy (what is called) the generalized Legendre-Clebsch necessary condition [23], i.e.

$$(-1)^p \frac{d^{2p}}{dt^{2p}} H_u \geq 0, \quad \forall t \in [t_1, t_2]. \quad (18)$$

The opposite inequality is valid for a maximization problem.

For problems with multiple inputs, (17) and (18) apply to each control input with some additional matrix necessary conditions for optimality; see [23, Thm 6.2 and Cor. 6.3]. We use this result to prove that there cannot be singular arcs in the problem considered here.

Proposition 1: For the optimal control problem (8), every feasible solution (x, λ, u) is regular, i.e. does not have singular arcs. That is, the optimal control contains only bang-bang arcs over $[t_0, t_f]$.

Proof: Assume the contrary, i.e. that the optimal control contains singular subarcs. This implies that one or more of the switching functions vanish over a nonempty open interval of some $(t_1, t_2) \subset [t_0, t_f]$. We look at each such possibility. From (10), the switching function vector is given as:

$$\begin{aligned} \sigma(x, \lambda, t) &:= \left[\frac{\partial H}{\partial u_1} \quad \frac{\partial H}{\partial u_2} \right]^T \\ &= \left[G(x_2 + \frac{\lambda_2}{M+\mu_{\infty}}) - B_{pto} x_2 (x_2 + \frac{\lambda_2}{M+\mu_{\infty}}) \right]^T, \quad (19) \\ &\quad \forall t \in [t_0, t_f]. \end{aligned}$$

The only two cases under which an optimal singular arc may be possible are:

i) $\sigma_1(x, \lambda, t) = 0$ over (t_1, t_2) . (This also implies $\sigma_2(\cdot) = 0$ over the same interval).

Assuming that the wave excitation force $f_{exc}(t)$ is sufficiently smooth, successive differentiations of $\sigma_1(\cdot)$ and substituting the state and adjoint dynamics (7) and (12), respectively, reveals that the minimum integer q that satisfies (17) is $q = 3$. This contradicts the fact such q should be even and, therefore, case (i) is not possible.

ii) $\sigma_2(x, \lambda, t) = 0$ over (t_1, t_2) and $\sigma_1(\cdot) \neq 0$ except possibly at a finite number of points over the same interval, i.e. $x_2 = 0$ with $\lambda_2 \neq 0$ except possibly at a finite number of points over (t_1, t_2) .

Taking a single time derivative of $\sigma_2(\cdot)$ reveals $\dot{x}_2 = 0 \forall t \in (t_1, t_2)$. This implies $u_1(t) = \frac{C_r x_{3,n}(t) + C x_1(t) - f_{exc}(t)}{G}$ over the same interval. However, since $\sigma_1(\cdot) \neq 0$ except possibly at a finite number of points over (t_1, t_2) , u_1 should only take its boundary values in this interval. Therefore, case (ii) is not possible and this proves our result. ■

C. Optimal Control Algorithm: a Gradient Projection Scheme

Although we had started Section III-A with the assumption that our control inputs can take a continuum of values in a bounded set, it has been shown that the optimal control inputs take values only on the boundary of the feasible set.

With this in mind and assuming the digital control implementation will be piecewise constant, we solve an approximate finite-dimensional optimization problem. The new problem is approximate in the sense that we are seeking an input in a piecewise continuous and bounded subset of the infinite dimensional original feasible set $\mathbb{U}$ but solve the same objective function as in OCP. We outline the online control synthesis algorithm below.

Let the optimization interval $[t_0, t_f]$ be divided into $N \in \mathbb{Z}^+$ equal subintervals and $h := \frac{t_f - t_0}{N}$ be the sampling period with sampling instants $t_{j+1} = t_j + h$ and piecewise constant control inputs, i.e. $u(t) = u(t_j) \in \mathbb{R}^m$, $\forall t \in [t_j, t_{j+1})$, $\forall j \in \{0, 1, \dots, N-1\}$. Let $u_{i,j} = u_i(t_j)$, then our aim is to find an optimal control sequence $\bar{u} = \{u_{1:m,j}\} \in \mathbb{V} \subset \mathbb{R}^{m \times N}$, where $\mathbb{V} = \{\{u_{1:m,j}\} : u_{i,j} \in [u_{\min,i}, u_{\max,i}] \subset \mathbb{R}, i = 1, \dots, m, j = 0, 1, \dots, N-1\}$, $u_{\min,i}$ and $u_{\max,i}$ are the lower and upper bounds, respectively, on the i^{th} control input and $u_{1:m,j}$ is to be interpreted as the input vector at time t_j .

Although more advanced schemes could be used (see Section IV), the method we adopt here is to iteratively improve the input sequence by minimizing the objective function (9) using a variation of the steepest descent method. This method has the nice property that it is globally convergent to stationary points under mild assumptions [24]. The main advantage of our particular scheme is its smaller computational cost; it requires only a single state and adjoint evaluation at each iteration and converges within a small number of iterations.

With a feasible initial choice $\bar{u}^0$, traditional gradient methods like the steepest descent method seek iterates

$$\hat{u}^{k+1} := \bar{u}^k - s^k \nabla J(\bar{u}^k), \quad (20)$$

where s^k is the step size at iteration k . The gradient, $\nabla J(\bar{u}^k) := \frac{dJ}{d\bar{u}}(\bar{u}^k)$, is an $m \times N$ matrix whose components in (21) measure the variation of the cost function with respect to each input and within each sampling interval. From (11) we get:

$$\begin{aligned} \nabla J(\bar{u}^k)_{i,j} &= \int_{t_j}^{t_{j+1}} \frac{\partial H^T}{\partial u_i}(x(t), u(t), \lambda(t), t) dt \big|_{\bar{u}^k}, \\ &= \int_{t_j}^{t_{j+1}} \sigma_i(x(t), \lambda(t); t) dt \big|_{\bar{u}^k}, \end{aligned} \quad (21)$$

where $i = 1, \dots, m$ refers to the input component and $j = 0, \dots, N-1$ identifies the sampling interval in $[t_0, t_f]$.

The tenet of a projected gradient method (PGM) is that it keeps the iterates feasible. In every iteration, a step in the direction of the anti-gradient is taken and the result in (20) is projected onto the feasible set $\mathbb{V}$,

$$\begin{aligned} \bar{u}^{k+1} &:= P_{\mathbb{V}}(\hat{u}^{k+1}), \\ P_{\mathbb{V}}(\hat{u}) &:= \arg \min_{\bar{v} \in \mathbb{V}} \|\bar{v} - \hat{u}\|. \end{aligned} \quad (22)$$

$$\text{Therefore, } \bar{u}_{i,j}^{k+1} = \begin{cases} u_{\max,i} & \text{if } \hat{u}_{i,j}^{k+1} > u_{\max,i}, \\ u_{\min,i} & \text{if } \hat{u}_{i,j}^{k+1} < u_{\min,i}, \\ \hat{u}_{i,j}^{k+1} & \text{otherwise,} \end{cases} \quad (23)$$

for $i = 1, \dots, m, j = 0, \dots, N-1$.

Although this projection operation can be computationally demanding with a substantial overhead for a general feasible set, it is easily computed for simple convex sets like the

polyhedron (or box) set $\mathbb{V}$ considered here. The projection is a simple element-wise bounding in (23) and, therefore, its computational demand is marginal. To make use of existing PGM results, we make the following technical assumptions:

Assumptions:

- 1) The objective function $J(\cdot; x_0)$ is continuously differentiable and bounded from below on the closed convex set $\mathbb{V}$.
- 2) The gradient $\nabla J(\cdot; x_0)$ is Lipschitz, i.e. $\exists L \geq 0$ such that $\|\nabla J(\bar{u}) - \nabla J(\bar{v})\| \leq L \|\bar{u} - \bar{v}\|$, $\forall \bar{u}, \bar{v} \in \mathbb{V}$, where $\|\cdot\|$ can be any p-norm. We assume, of course, that x_0 is bounded and that the state and co-state trajectories stay bounded.

From the definitions of the dynamics and the objective function, it is trivial that the cost $J(\cdot)$ is continuous in the input and bounded from below. Using the standard result that the state and adjoint variables are continuous even under piece-wise continuous (or bang-bang) inputs [21], from (21), the gradient of the cost is continuous with respect to input variations in $\mathbb{V}$. The Lipschitz assumption results from the continuity and boundedness of $\nabla J(\cdot)$ over the compact set $\mathbb{V}$. This could also be inferred from the physical principle that we could not extract infinitely more energy from the device using a finite change in the control forces.

With these assumptions, one can show that the projected gradient method converges to a local minimum for various step-size rules [24, Sec. 2.3]. As in the steepest descent method, the limitation of this method is that it generally has poor convergence. Nonetheless, it is shown in [24] that fast (superlinear) convergence can be achieved using a combination of Armijo-type line search schemes and Newton and quasi-Newton methods. These, however, are complex algorithms performing line searches and associated function and Hessian evaluations at each iteration and have overheads comparable to complex NLP solvers. For convex problems, fast gradient methods that use a constant step-length with slight modifications can still achieve the best of either linear or quadratic convergence rates [25]. Since the problem considered here is not convex and the aim is to avoid the overhead incurred in performing line searches and associated Hessian and function evaluations at each step, we opt for a constant step-length scheme.

Theorem 2: With the above assumptions satisfied on the closed convex set $\mathbb{V}$, taking a constant step-size $s^k = s$, where $0 < s \leq \frac{2(1-\sigma)}{L}$, $0 < \sigma < 1$, the algorithm in (22) is globally convergent to a local minimum (or stationary point) and the following are valid:

$$J(\bar{u}^k) - J(\bar{u}^{k+1}) \geq \frac{\sigma}{s} \|\bar{u}^k - \bar{u}^{k+1}\|^2, \quad (24a)$$

$$J(\bar{u}^k) - J(\bar{u}^{k+1}) \geq \min\{1, s\} \frac{\sigma}{s} \|\bar{u}^k - P_{\mathbb{V}}\{\bar{u}^k - s \frac{dJ^T}{d\bar{u}}(\bar{u}^k)\}\|^2. \quad (24b)$$

Proof: The global convergence of the algorithm and the conditions in (24a)-(24b) are a standard result and relegated to [24, Sec. 2.3.2] or [26, Thm 4.1]. ■

We have proposed a method that is globally convergent to a local minimum. Although only a sublinear global convergence rate can be guaranteed, the algorithm converges much quicker (at worst, within a few tens of iterations) than what is predicted

by the convergence analysis – the reduction in the cost at each iteration in (24b) does not diminish until we get very close to a minimum, in which case we have converged for all practical purposes and the solution can be rounded to the nearest vertex of the polyhedron set $\mathbb{V}$. It should also be noted that the Lipschitz constant L is usually unavailable, and so we cannot determine the range of feasible step sizes s^k a priori. We will, however, demonstrate that choosing an appropriate s^k via offline simulations under various conditions would suffice.

IV. EXAMPLE SIMULATIONS

For the semi-submerged heaving cylinder that we consider, non-dimensionalized impulse response kernels for the radiation and excitation forces from [18] were used. We scale the problem to an appropriate size roughly comparable to the device in [6]; a cylinder of radius $R = 5$ m, and 20 m high with a spring of stiffness $k = 240$ kN/m will be used. From the dimensionalization relation used in [18], we calculate the draught to be 9 m in 42.85 m deep waters. The spring is assumed slack at equilibrium (no wave), and the mass of the device is $M = 707$ t (tonnes) with $\mu_\infty = 0.345 * M \approx 244$ t from the relation in [18] and water density $\rho = 1000$ kg/m³.

Fig. 2 shows that a 3rd order radiation subsystem is enough to approximate the sampled radiation impulse response. We also generate the excitation force using a 6th order state space model approximating “causalized” excitation impulse response and driven by the wave height data at the buoy [18].

A. Projected Gradient Algorithm Performance

The projected gradient method was tested to see its convergence properties under different wave conditions. The results were compared to the results from solving a direct transcription of the optimal control problem using a state of the art open-source optimization software (IPOPT, version 3.9.2) [27]. Euler, trapezoidal and Hermite-Simpson collocation schemes were used for the IPOPT implementation (see [28, Sec. 4.5]). The explicit Runge-Kutta (4,5) and a variable order solver based on numerical differentiation formulas (Matlab’s `ode45` and `ode15s`, respectively,) were used to integrate the dynamics and adjoint dynamics within the PGM implementation. Our implementation in IPOPT adds the input constraints to the objective function using barrier functions to form a Lagrangian. At each iteration, gradient and Hessian computations of the Lagrangian, as well as a number of line searches are performed (see [24]). On the other hand, the PGM method described in Section III-C requires only a single state and adjoint state evaluation; the gradient computation and the projection onto the feasible input set add only marginal computational cost.

In all the simulations, the control inputs have a sampling period of 0.1s; the states and adjoint states are resolved at a 5 times finer rate. IPOPT was set to use an adaptive barrier parameter update strategy since it resulted in better convergence in simulations. For the PGM, a value for the constant step-size was chosen a posteriori from simulations. A value of $s = \frac{2}{G}$ was found to work sufficiently well under all the wave conditions presented. It can be seen from Fig. 3 that the PGM converges more quickly than IPOPT to the

same local optima. The test was done under different wave conditions, parameter values and WEC initial conditions to confirm similar performances. A snippet of the device response under the controller is shown in Fig. 3(c).

B. Device Optimization for Control

In addition to being optimized for a given wave climate, the parameters of a PTO should be designed with a specific control scheme in mind; it is shown here that a design that is optimal under one control scheme may not be optimal under a different control scheme. For example, [29] shows that, under a given sea condition, the optimal damping coefficient of a generic point absorber is very different depending on whether latching control is used or the device is uncontrolled. We have made simulations to choose the control parameters B_{pto} and G in an optimal way for a given wave environment and control scheme. Fig. 4(a) shows a sweep of these parameter values against average power delivered when the active controller of the previous section is used over 50 s prediction horizons; here the computed controls are applied over the whole 50 s. The simulations were carried out over 1000 seconds with a wave from a JONSWAP spectrum of typical period $T_p = 8$ s.

The simulation indicates that energy yield increases with both parameters until it flattens. Similarly to the control scheme discussed in Section III-A, we consider an actuation mechanism where the damper of the PTO is not controlled and is always on. A similar optimal control problem formulation and numerical scheme was used; only one input is considered, i.e. the active control input u_1 ; $u_2(t) = 1, \forall t$. The resulting optimal scheme can be shown to be bang-bang. Fig. 4(b) shows a parameter sweep for this control scheme. The optimal damper value is $B_{pto} \approx 280$ kN·s/m for the range of G shown. Here the curve is concave and further increasing the damping coefficient decreases yield, unlike in the case where the damping element is controlled. It seems that higher values of a damper element do not result in higher yield unless the element is controlled. This, perhaps, explains why the optimal damping coefficients in declutching control tend to be much higher than ones for latching control (as is apparent from results in [30]). Thus, the optimization of PTO parameters should heavily depend on the control scheme intended.

We also compare the control schemes developed with optimal latching control and the uncontrolled system. We will label the active control scheme with control over both the damping and active PTO elements ‘Method 1’ and the case where the linear damper is always engaged ‘Method 2’. The optimal command latching scheme employed is exactly as in [6]. A similar parameter optimization for the latching control revealed that $B_{pto} \approx 95.1$ kN·s/m gives the best results under the same wave conditions stated. The absorbed power function in Fig. 4(c) shows both active controllers and the optimal latching result in an increase in extracted energy compared to the hydraulic PTO with no control. As expected, the latching controller enlarges the bandwidth only towards low frequencies. Unlike latching control, the active methods widen the bandwidth of the WEC in both directions around the resonant frequency; latching is effective only at frequencies lower than that of the buoy [6].

Fig. 2. (a) A radiation subsystem of order 3 is good enough to approximate the radiation impulse response sufficiently. (b) A 6th order subsystem can be used to model the “causalized” excitation force.

Fig. 3. Convergence of the PGM algorithm (dotted, left) against that of a direct transcription solution using IPOPT (lines with marker) over a 50 s prediction horizon (a) for waves of the same typical period (T_p) and different significant wave heights (H_s) (b) for waves of different typical period (T_p) but of the same significant wave height. (c) A plot of the device velocity against control commands and normalized excitation force under ‘Method 1’

Fig. 4. Variations in average absorbed power (W) against parameters G and B_{pto} for: (a) Method 1 (b) Method 2; (c) Average absorbed power against typical wave period with the different control methods, $B_{pto} \approx 280 \text{ kN}\cdot\text{s/m}$ ($B_{pto} \approx 95 \text{ kN}\cdot\text{s/m}$ for latching control), $G = M + \mu_\infty$, $T_p = 8 \text{ s}$, $H_s = 2\text{m}$.

Method 1 also shows consistently better performance compared with Method 2. Since linear dampers can be switched on and off using a simple by-pass valve, optimally controlling passive PTO elements for better performance can be justified.

C. Prediction Horizon Sensitivity

In the preceding discussion, a 50 s prediction horizon was used for device parameter optimization under ideal conditions where the whole predicted control is applied — no disturbances were applied and the wave height assumed known. In a real implementation, optimal controllers are computed over a prediction horizon and then only part of the control is applied over what is called the control interval; this can be one sampling period but often much longer (many multiples of the sampling time depending on the application). Here, we investigate the performance of the method against varying prediction and control horizons.

Fig. 5 shows that the power output increases with prediction horizon length until it flattens around three times the typical period value. We also show the use of different control intervals (T_c). Although decreasing T_c gives better performance in general, its effect can be negligible for large prediction horizons; compare the 20kW increase in power when T_c is halved at $T_p = 3\text{s}$ with the 0.1 kW difference at $T_p = 8\text{s}$. The integration of multi-step wave excitation prediction schemes in this sensitivity analysis, as done using an extended Kalman filter in [13], can also be considered.

V. CONCLUSION

In this paper, an optimal active control method for a receding horizon control strategy was considered. A state space model of a generic point absorber, whose power take-off includes a linear damper and an active element, was formulated and used. By considering a variational formulation of the optimal control problem, the solution was shown to be a bang-bang type when

Fig. 5. Variations in average absorbed power against prediction horizon lengths for 'Method 1'. JONSWAP wave with $T_p = 8$ s and $H_s = 2$ m used; $G = B = 0.3 * (M + \mu_\infty)$.

constraints are imposed only on the control forces. Although we have considered an axisymmetric device oscillating in heave only, the analysis can be generalized to devices moving in many degrees of freedom.

A computationally inexpensive and globally convergent numerical scheme was developed for solving the power maximization problem. A variation of the projected gradient method (PGM) was exploited and shown to converge in few iterations under various wave conditions. Its performance has been compared to solving a directly collocated version of the problem using a state of the art interior point solver, IPOPT. As the PGM requires only a single state and adjoint state evaluation at each iteration, it was shown to be far less computationally demanding compared to a general NLP solver. Time-domain simulations have also been used to evaluate the performance of the controllers developed. The optimization of PTO system parameters was shown to be vital and highly dependent on the control scheme used.

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