a singular distribution. In particular, if the input signal is predictable, it was shown that the output of the filter is simply a discrete random variable. Computer experiments in order to verify the theoretical results have been realized and discussed. For this purpose, a specific model of linear Markovian signal of order one was introduced, and the experimental results are in complete agreement with the theory. Finally, a model of Bernoulli input ensuring that the output contains a discrete and a singular part was introduced, and here also the experimental results are in perfect agreement with the theory.

REFERENCES

- [1] B. Picinbono and J.-Y. Tournet, "Singular random signals," *IEEE Trans. Signal Process.*, vol. 53, no. 2, pp. 499–504, Feb. 2005.
- [2] B. Picinbono, *Random Signals and Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
- [3] E. Lukacs, *Characteristic Functions*. London, U.K.: Griffin, 1970.
- [4] P. Wittke, W. Smith, and L. Campbell, "Infinite series of interference variables with Cantor-type distributions," *IEEE Trans. Inf. Theory*, vol. 34, pp. 1428–1436, Nov. 1988.
- [5] M. Naraghi-Pour and D. Neuhoff, "On the continuity of the stationary state distribution of DPCM," *IEEE Trans. Inf. Theory*, vol. 36, pp. 305–311, Mar. 1990.
- [6] W. Smith, P. Wittke, and L. Campbell, "Error probabilities on fading channels with intersymbol interference and noise," *IEEE Trans. Inf. Theory*, vol. 39, pp. 1598–1607, Sep. 1993.

Statistical Properties of the Estimated Degree of Polarization

T. Medkour and A. T. Walden, *Member, IEEE*

Abstract—We derive important and useful new statistical properties of the estimated degree of polarization in the two-dimensional case. We find its distribution function and show how it may be used to construct confidence intervals. We also find an expression for any moment of the distribution, and derive an exact unbiased formula for the estimator of the squared coefficient. Further we discuss a test for partial polarization. Our statistical analyses enable identification of elliptical polarization for an ultra low frequency wave in the solar magnetic field.

Index Terms—Confidence intervals, cumulative distribution, degree of polarization, moments, spectrum estimation.

I. INTRODUCTION

Studies of the polarization states of vector processes have been, and continue to be, very widely carried out and discussed in radar research [24], atmospheric science [11], geophysics [18], [19], oceanography [5], and optics [3], [4], [9], [12], [21]. A key quantity in these studies is the degree of polarization (DoP), definitions of which vary for dimensionality exceeding two [21].

Manuscript received November 23, 2006; revised June 6, 2007. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Antonio Napolitano. The work of T. Medkour is supported in part by the government of the People's Democratic Republic of Algeria.

The authors are with the Department of Mathematics, Imperial College London, London SW7 2BZ, U.K. (e-mail: tarek.medkour@imperial.ac.uk; a.walden@imperial.ac.uk).

Digital Object Identifier 10.1109/TSP.2007.906767

In studies of planar wave fields propagating in the z direction with the wave field oscillating in the x - y plane, the DoP—the ratio of the power of the polarized part of the wave to its total power—may be readily and unambiguously defined [4].

In this correspondence, we concentrate on this two-dimensional case and consider the statistical properties of the estimated DoP found from a sample Hermitian covariance matrix of the form

$$\begin{aligned}\hat{\Sigma} &= \frac{1}{K} \sum_{k=0}^{K-1} \begin{bmatrix} Z_{1,k} Z_{1,k}^* & Z_{1,k} Z_{2,k}^* \\ Z_{1,k}^* Z_{2,k} & Z_{2,k} Z_{2,k}^* \end{bmatrix} \\ &= (1/K) \sum_{k=0}^{K-1} \mathbf{Z}_k \mathbf{Z}_k^H, \end{aligned} \quad (1)$$

where the averaging is over K independent bivariate samples $\mathbf{Z}_k = [Z_{1,k}, Z_{2,k}]^T$ derived from the wavefield, "*" denotes complex conjugate, and the samples have the proper complex bivariate Gaussian distribution with zero-mean and positive-definite covariance matrix $E\{\mathbf{Z}_k \mathbf{Z}_k^H\} = \Sigma$. Of course, as $K \rightarrow \infty$, so $\hat{\Sigma} \rightarrow \Sigma$. If ς_1 and ς_2 are the eigenvalues of Σ , ($\varsigma_2 \leq \varsigma_1$), then the true DoP, P , can be written as $P = (\varsigma_1 - \varsigma_2)/(\varsigma_1 + \varsigma_2)$ [4], [9]. Partial polarization occurs for $0 < P < 1$. $\text{Im}\{\Sigma_{12}\} = 0$ (where Im denotes imaginary part) corresponds to rectilinear polarization. $\text{Im}\{\Sigma_{12}\} \neq 0$ corresponds to elliptical polarization and if $\Sigma_{11} = \Sigma_{22}$ also, the polarization is circular [11].

Synthetic aperture radar (SAR) measurements of the Earth's surface provide important insights into the nature of backscattering from natural targets. The data are subject to speckle, and some form of averaging is needed. The multi-look SAR covariance matrix has the form $\hat{\Sigma}$, with K being the number of looks [24].

Seismic waveforms are typically superpositions of direct, refracted, reflected, and scattered waves, and polarization is unlikely to be constant with frequency, but a well-defined polarization structure can nevertheless arise by the coherent addition of scattered waves within the crustal waveguide [16]. Samson [18] gives a method of estimating polarization as a function of frequency; $\hat{\Sigma}$ takes the form of an estimated Hermitian spectral matrix computed with K complex degrees of freedom from two jointly-stationary real-valued processes $\{U_t\}$ and $\{V_t\}$; details are given in Section V.

In practice, K will be finite, and consequently P must be estimated as $\hat{P}$ via the eigenvalues of the estimated covariance matrix, $\hat{\Sigma}$, in (1). In this correspondence, we study the statistical properties of the estimated DoP $\hat{P}$. A statistical study of a different quantity, the estimated degree of linear polarization was given in [23] (see, e.g., [12] for definitions of terms such as the degree of linear polarization and the degree of circular polarization).

Given the statistical structure assumed for the components of $\hat{\Sigma}$ in (1), the random matrix has the complex Wishart distribution (see Section II). We derive a finite sum expression for the statistical distribution function of $\hat{P}$ in Section II and show how this enables us to set confidence intervals for the true DoP P . Moments of order r are found, and the resulting mean and standard deviation are confirmed by simulation. Since bias is such an important factor, in Section III we derive an exactly unbiased estimator for P^2 and compare our estimator with that of Samson [18] via simulation. A statistical test for partial polarization is given in Section IV and related to sphericity tests. A case-study using our techniques is provided in Section V based on solar magnetic field data. Concluding comments are provided in Section VI.

II. STATISTICAL FRAMEWORK

A. The Eigenvalues

If $\mathbf{Z}_0 = [Z_{1,0}, Z_{2,0}]^T$ has the proper complex bivariate Gaussian distribution with zero mean and covariance matrix $E\{\mathbf{Z}_0 \mathbf{Z}_0^H\} = \mathbf{\Sigma}$, then it has probability density function (PDF)

$$f_{\mathbf{Z}_0}(z) = \pi^{-2} \det^{-1}(\mathbf{\Sigma}) \exp(-z^H \mathbf{\Sigma}^{-1} z), \quad z \in \mathcal{C}^2 \quad (2)$$

where “ H ” denotes Hermitian transpose. We assume $\det\{\mathbf{\Sigma}\} > 0$ in order that the PDF exists. So the statistical development is valid for a DoP strictly less than unity. Suppose $\mathbf{Z}_0, \dots, \mathbf{Z}_{K-1}$ are a random sample of K such vectors.

Let $\mathbf{Z} = [\mathbf{Z}_0, \mathbf{Z}_1, \dots, \mathbf{Z}_{K-1}]$ and consider the random 2×2 matrix $\mathbf{Z} \mathbf{Z}^H = \sum_{k=0}^{K-1} \mathbf{Z}_k \mathbf{Z}_k^H$. When $K \geq 2$, the matrix is full-rank and $\mathbf{W} = \sum_{k=0}^{K-1} \mathbf{Z}_k \mathbf{Z}_k^H$ has the nonsingular 2-D complex central Wishart distribution with K complex degrees of freedom and mean $K \mathbf{\Sigma}$, [8]. Let ω_1 and ω_2 denote the eigenvalues of $\mathbf{W}$, where $\omega_2 < \omega_1$. $\hat{\mathbf{\Sigma}} = \mathbf{W}/K$ given in (1) is the maximum-likelihood estimator of the true covariance matrix $\mathbf{\Sigma}$ and has eigenvalues ω_1/K and ω_2/K .

B. Distribution of the Degree of Polarization

The estimated DoP is

$$\hat{P} = \frac{(\omega_1/K) - (\omega_2/K)}{(\omega_1/K) + (\omega_2/K)} = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \quad (3)$$

$0 < \omega_2 \leq \omega_1 < \infty$. So $\hat{P} = (\omega_1 - \omega_2)/(\omega_1 + \omega_2) = \hat{P}_{\mathbf{W}}$, say, the DoP corresponding to the random matrix $\mathbf{W}$, and so the statistical properties of $\hat{P}$ and $\hat{P}_{\mathbf{W}}$ are identical.

The joint distribution of (ω_1, ω_2) , $0 < \omega_2 \leq \omega_1 < \infty$ was given by Sikaneta and Chouinard [22]

$$f(\omega_1, \omega_2) = \frac{\omega_1^{K-1} \omega_2^{K-2} - \omega_1^{K-2} \omega_2^{K-1}}{\Gamma(K) \Gamma(K-1) (\varsigma_1 - \varsigma_2) (\varsigma_1 \varsigma_2)^{K-1}} \times [e^{-\omega_1/\varsigma_1 - \omega_2/\varsigma_2} - e^{-\omega_1/\varsigma_2 - \omega_2/\varsigma_1}]. \quad (4)$$

Given (4), we first find the joint distribution $f(p, d)$ of $\hat{P}$ (i.e., $\hat{P}_{\mathbf{W}}$) and $\hat{D} = \omega_1 - \omega_2$. For this transformation, $|J| = d/(2p^2)$, where J denotes the Jacobian. We find

$$f(p, d) = C^{-1} (1 - p^2)^{K-2} (d/p)^{2K-2} (e^{-(g-h)d} - e^{-(g+h)d})$$

where $C = 2^{(2K-3)} \Gamma(K) \Gamma(K-1) (\varsigma_1 - \varsigma_2) (\varsigma_1 \varsigma_2)^{K-1}$, and

$$g = \frac{\varsigma_1 + \varsigma_2}{2\varsigma_1 \varsigma_2} \frac{1}{p} > 0; \\ h = \frac{\varsigma_1 + \varsigma_2}{2\varsigma_1 \varsigma_2} P > 0.$$

We can simplify $f(p, d)$ to

$$f(p, d) = 2C^{-1} (1 - p^2)^{K-2} (d/p)^{2K-2} e^{-gd} \sinh(hd).$$

Integration over d from 0 to ∞ using [10, 3.5.51], and use of $(\varsigma_1 \varsigma_2)^K / (\varsigma_1 + \varsigma_2)^{2K} = [(1 - P^2)/4]^K$, yields

$$f(p) = \frac{4^{1-K} (1 - P^2)^K p (1 - p^2)^{K-2}}{B(K, K-1) P} \times [(1 - pP)^{1-2K} - (1 + pP)^{1-2K}], \quad (5)$$

$0 \leq p < 1$, where $B(a, b)$ is the beta function. The PDF (5) is the only analytical result on the DoP given recently in [20], where it was derived using a very different approach. The key step we take in obtaining more powerful analytical results such as confidence intervals from

Fig. 1. (a) PDF $f(p)$ and (b) CDF $F(p)$, for $K = 5$ and for $P = 0, 0.5, 0.8$, shown as solid, dashed, and dotted lines, respectively.

expanding the terms in the square brackets [1, 15.1.10] so that the PDF can be rewritten as (with dependence on K and P shown explicitly)

$$f(p; K, P) = \frac{2}{B(K-1, \frac{3}{2})} (1 - P^2)^K p^2 \times (1 - p^2)^{K-2} {}_2F_1\left(K, K + \frac{1}{2}; \frac{3}{2}; P^2 p^2\right) \quad (6)$$

where ${}_2F_1(\alpha_1, \alpha_2; \beta_1; z)$ is the hypergeometric function, a special case of the generalized hypergeometric series ${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$ defined by [10, p. 1045]

$$\sum_{m=0}^{\infty} \frac{(\alpha_1)_m \dots (\alpha_p)_m}{(\beta_1)_m \dots (\beta_q)_m} \frac{z^m}{m!} \quad (7)$$

where $(y)_m$ is defined in terms of the Gamma function $\Gamma(\cdot)$ as $(y)_m = y(y+1) \dots (y+m-1) = \Gamma(y+m)/\Gamma(y)$. $f(p)$ is plotted in Fig. 1(a) for $K = 5$ and true DoP values of $P = 0, 0.5, 0.8$.

In Appendix A, we find the very useful result that the cumulative distribution function (CDF) $F(p) = \Pr(\hat{P} \leq p)$ may be written as a *finite* sum of hypergeometric functions. It takes the form (again with dependence on K and P shown explicitly)

$$F(p; K, P) = (1 - P^2)^K p^3 \sum_{m=0}^{K-2} \frac{\Gamma(m + \frac{3}{2})}{\Gamma(\frac{3}{2}) m!} (1 - p^2)^m \times {}_2F_1\left(K, m + \frac{3}{2}; \frac{3}{2}; P^2 p^2\right). \quad (8)$$

Knowledge of the CDF enables the setting of confidence intervals for P . $F(p)$ is plotted in Fig. 1(b).

C. Confidence Intervals for P

If we define points $a_{\alpha/2}(P)$ and $a_{1-\alpha/2}(P)$ such that $F(a_{\alpha/2}(P)) = \alpha/2$ and $F(a_{1-\alpha/2}(P)) = 1 - \alpha/2$, then $\Pr[a_{\alpha/2}(P) \leq \hat{P} \leq a_{1-\alpha/2}(P)] = 1 - \alpha$. Examples of 90%

Fig. 2. 90% confidence regions for DoP when (a) $K = 10$ and (b) $K = 50$. The horizontal lines mark $p = 0.7$ and the intervals defined by the vertical dashed lines are the 90% confidence intervals for P .

intervals ($\alpha = 0.1$) are given in Fig. 2(a) and (b) for $K = 10$ and $K = 50$, respectively. Also graphically illustrated is a $(1 - \alpha)100\%$ confidence interval for P given an estimate p . This can be found by drawing a line across the plot at p ; the intersection of this line with $a_{1-\alpha/2}$ at P_1 say, and with $a_{\alpha/2}(P)$ at P_2 say, defines a $(1 - \alpha)100\%$ confidence interval for P as $[P_1, P_2]$, which depends on the K used. In the examples, $p = 0.7$ and the resulting 90% confidence intervals for P are $[0.35, 0.83]$ for $K = 10$ and $[0.60, 0.77]$ for $K = 50$. Interval widths narrow with increasing averaging K , as expected. In practice, given an estimate p of P , the right end of the interval P_2 is the value of P such that $F(p; K, P) - \alpha/2 = 0$, which can be found simply using any standard zero-finding algorithm. P_1 is likewise found by solving $F(p; K, P) - (1 - \alpha/2) = 0$.

D. Moments of the Distribution

The r th moment of $\hat{P}$ is given by $E\{\hat{P}^r\} = \int_0^1 p^r f(p) dp$, i.e.,

$$E\{\hat{P}^r\} = \frac{2}{B(K-1, \frac{3}{2})} (1-P^2)^K \int_0^1 p^{r+2} (1-p^2)^{K-2} {}_2F_1\left(K, K + \frac{1}{2}; \frac{3}{2}; P^2 p^2\right) dp$$

which, using [10, 3.251(1)], may be reduced to

$$E\{\hat{P}^r\} = \frac{B(K-1, \frac{r+3}{2})}{B(K-1, \frac{3}{2})} (1-P^2)^K \times {}_3F_2\left(K, K + \frac{1}{2}, \frac{r+3}{2}; \frac{3}{2}, K + \frac{r+1}{2}; P^2\right). \quad (9)$$

(The generalized hypergeometric series is defined in (7).) Two important properties of $\hat{P}$, namely the bias, $E\{\hat{P}\} - P$, and mean-squared error $E\{(\hat{P} - P)^2\}$ (variance plus squared bias), are shown in Fig. 3 for $K = 5, 10$, and 50 . Both decrease in a similar way with increasing P .

E. Simulation Results

For our simulations, we used

$$\Sigma = \begin{bmatrix} 10 & -1 + 2i \\ -1 - 2i & 3 \end{bmatrix}$$

for which $P = 0.639$. We simulated $\hat{\Sigma} = (1/K) \sum_{k=0}^{K-1} \mathbf{Z}_k \mathbf{Z}_k^H$ by simulating bivariate complex Gaussian $\mathbf{Z}$ using the method in [13, Sec. V]. For $K = 3, 6, 10$, and 50 , we independently simulated 1000 realizations of $\hat{\Sigma}$, and consequently 1000 realizations of (ω_1, ω_2) . Com-

Fig. 3. (a) Bias and (b) mean-squared error of $\hat{P}$ as a function of the true value P for $K = 5, 10$, and 50 , shown as solid, dashed and dotted lines, respectively.

TABLE I
RESULTS OF THE SIMULATION STUDY. $P = 0.639$ AND $P^2 = 0.408$.
SEE TEXT OF SECTIONS II-E AND III-B AND FOR NOTATION DETAILS

K	$E\{\hat{P}\}$	$\bar{P}$	$\sigma\{\hat{P}\}$	$s\{P\}$	$\bar{Q}$	$\bar{R}$
3	0.760	0.760	0.166	0.167	0.406	0.501
6	0.694	0.695	0.146	0.145	0.411	0.434
10	0.669	0.668	0.122	0.121	0.411	0.419
50	0.645	0.645	0.059	0.058	0.410	0.410

pared in the left part of Table I are the sample averages $\bar{P}$ and sample standard deviation $s\{P\}$ of $\hat{P}$ over the realizations, with the theoretical mean $E\{\hat{P}\}$ and standard deviation $\sigma\{\hat{P}\}$. The agreement is excellent, supporting (9).

III. UNBIASED ESTIMATION OF P^2

A. Exactly Unbiased Estimator

While most researchers (e.g., [9] and [12]) consider P to be the basic quantity of interest, others define P^2 to be “the degree of polarization” [18], [19]. Indeed, Samson [18] produced approximate corrections for the sample bias in $\hat{P}^2$ as an estimator of P^2 . In this section, we consider an exactly unbiased estimator $\hat{Q}$ for P^2 such that $E\{\hat{Q}\} = P^2$.

Let $G(\hat{P})$ be the unbiased estimator such that $E\{G(\hat{P})\} = 1 - P^2$. We first find the unbiased estimator for $1 - P^2$, then the unbiased estimator we want is $\hat{Q} = 1 - G(\hat{P})$, [15] since $E\{\hat{Q}\} = 1 - E\{G(\hat{P})\} = 1 - (1 - P^2) = P^2$. We show in Appendix B that $G(\hat{P}) = (1 - \hat{P}^2) {}_2F_1(3/2, 1; K; 1 - \hat{P}^2)$, so our *exactly* unbiased estimator for P^2 is given by

$$\hat{Q} = 1 - (1 - \hat{P}^2) {}_2F_1\left(\frac{3}{2}, 1; K; 1 - \hat{P}^2\right). \quad (10)$$

(It does not appear possible to find an exactly unbiased estimator for P rather than P^2 .) Samson [18, eq. 39] used a Taylor series expansion

Fig. 4. Critical value C as a function of K for $\alpha = 0.05$ (solid) and 0.01 (dashed).

to derive an *approximately*—to first order in $1/K$ —unbiased estimator for P^2 , as

$$\hat{R} = \hat{P}^2 + (2/K) \left[4 \frac{\omega_1^3 + \omega_2^3}{(\omega_1 + \omega_2)^3} - 3 \frac{(\omega_1^2 + \omega_2^2)^2}{(\omega_1 + \omega_2)^4} - 1 \right]. \quad (11)$$

Both the estimators $\hat{Q}$ and $\hat{R}$ can produce negative estimates. Hence, in practice $\max\{0, \hat{Q}\}$ and $\max\{0, \hat{R}\}$ would be used.

B. Simulation Results

The sample averages $\bar{Q}$ and $\bar{R}$ of $\hat{Q}$ and $\hat{R}$ over the realizations generated as in Section II-E are given on the right side of Table I. For this model, $P^2 = 0.408$. It is verified that whereas $\hat{Q}$ performs equally well for all K , $\hat{R}$ is poor for small K but improves as K increases. It is important that the estimator works well for small K since in the context of (1), a small value of K is quite common in practice.

IV. TESTING FOR PARTIAL POLARIZATION

A. Test Statistic and Its Distribution

Suppose we wish to test the hypothesis $H_0 : P = 0$ versus $H_1 : 0 < P$. When $P = 0$, the PDF of $\hat{P}$, (6) takes the simple form $f(p) = [2/B(K-1, 3/2)]p^2(1-p^2)^{K-2}$. This PDF is shown in Fig. 1(a) by the solid line for $K = 5$. The PDF of $A = \hat{P}^2$ is then given by

$$f(a) = \frac{1}{B(K-1, \frac{3}{2})} a^{1/2} (1-a)^{K-2} \quad (12)$$

so that $A = \hat{P}^2$ has a beta distribution with parameters $3/2$ and $K-1$. Therefore

$$\begin{aligned} \Pr(\hat{P}^2 \leq x) &= \int_0^x \frac{1}{B(K-1, \frac{3}{2})} a^{1/2} (1-a)^{K-2} da \\ &= I_x(3/2, K-1) = \Pr(\hat{P} \leq x^{1/2}) \end{aligned}$$

where $I_x(b, c)$ is the incomplete beta function. Hence, $\Pr(\hat{P} \leq x) = I_x(3/2, K-1)$ and if $\alpha = \Pr(\hat{P} > C)$, then $\alpha = 1 - I_{C^2}(3/2, K-1)$ so that $C : \{I_{C^2}(3/2, K-1) = 1 - \alpha\}$ is the critical value for the hypothesis test of size α . These critical values are plotted in Fig. 4 for $K = 2$ to 100 , and for $\alpha = 0.05$ and 0.01 . It is sobering to see how large $\hat{P}$ should be before we reject $P = 0$ when K is small. For example, for $K = 5$, the critical value is 0.78 , which is consistent with Fig. 1(a) which shows the distribution of $\hat{P}$, when $P = 0$, stretches across from 0 to 1 .

B. Relationship to Sphericity Testing

Since $P = 0$ corresponds to $\Sigma_{11} = \Sigma_{22} = \sigma^2$, say, and $\Sigma_{12} = \Sigma_{21} = 0$, the test for partial polarization is the same as a statistical test for sphericity under the null hypothesis $\Sigma = \sigma^2 \mathbf{I}$. Under this null hypothesis, the distribution of $T \equiv 4\det\{\Sigma\}/\text{tr}^2\{\Sigma\}$ was derived in [14]. But another way of writing P is as [4]

$$P = \left[1 - \frac{4\det\{\Sigma\}}{\text{tr}^2\{\Sigma\}} \right]^{1/2} \quad (13)$$

so that $A = 1 - T$, and indeed the distributions of A in (12) and $1 - T$ are the same, as required.

Under more general assumptions about Σ , such as considered in this correspondence, the distribution of T , and hence by implication $A = \hat{P}^2$, was determined in [17], but the distribution is there given in terms of zonal polynomials and Meijer's G-function.

V. EXAMPLE: FREQUENCY-DEPENDENT POLARIZATION

In the Introduction it was pointed out that in geophysical applications $\hat{\Sigma}$ can take the form of an estimated spectral matrix at a frequency f say. Consider jointly stationary real-valued time series $\{U_t\}$ and $\{V_t\}$ in the x and y directions, respectively.

As a K -sample estimator of the spectral matrix $S(f)$ for these series, we take the multitaper estimator [25]

$$\begin{aligned} \hat{S}(f) &= \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{J}_k(f) \mathbf{J}_k^H(f) \\ &= \begin{bmatrix} \hat{S}_{UU}(f) & \hat{S}_{UV}(f) \\ \hat{S}_{UV}^*(f) & \hat{S}_{VV}(f) \end{bmatrix} \end{aligned} \quad (14)$$

where $J_{U,k}(f) \equiv (\Delta t)^{1/2} \sum_{t=0}^{N-1} h_{k,t} U_t e^{-i2\pi f t \Delta t}$, and likewise for $J_{V,k}(f)$, and $\mathbf{J}_k(f) = [J_{U,k}(f), J_{V,k}(f)]^T$. Here, Δt is the sample interval, and $\{h_{k,t}\}$ is the k th-order orthonormal taper, where, by convention, $k = 0, \dots, K-1$. Then (e.g., [25]), asymptotically, $\mathbf{J}_k(f)$ has a complex bivariate Gaussian distribution with mean $\mathbf{0}$ and covariance matrix $S(f)$ for $0 < |f| < f_N$, where $f_N = 1/(2\Delta t)$. Multitapering has two benefits. The tapering reduces sidelobe leakage when there is a high dynamic range to the spectrum around the frequency of interest (as in the application here) and the orthogonal tapers provide K random vectors $\mathbf{J}_k(f)$ having the same (large-sample) statistical properties as $\mathbf{Z}_k$ in (1).

Then, (14) is the statistical equivalent of (1) and the results derived for $\hat{P}$ will apply for $\hat{P}(f) = [\hat{\lambda}_1(f) - \hat{\lambda}_2(f)]/[\hat{\lambda}_1(f) + \hat{\lambda}_2(f)]$, where $\hat{\lambda}_1(f)$ and $\hat{\lambda}_2(f)$ are the eigenvalues of the estimated spectral matrix $\hat{S}(f)$.

We shall examine solar magnetic field data recorded by one of the spacecraft in the Cluster mission in February 2003. This mission is an international solar physics experiment to collect data on various aspects of the Sun. Fig. 5 shows two time series resulting from the presence of an ultra-low-frequency (ULF) wave in the solar magnetic field [2]. The measurement unit is nanoTeslas (nT), and the sample interval is $\Delta t = 2$ s, giving a Nyquist frequency of $1/(2\Delta t) = 0.25$ Hz. The solid line is the component $\{U_t\}$, measured parallel to the ecliptic plane, and the dotted line is the component $\{V_t\}$, measured perpendicular to the ecliptic plane. Fig. 6(a) gives the estimated spectra $\hat{S}_{UU}(f)$ and $\hat{S}_{VV}(f)$ computed using six Slepian tapers with an effective analysis bandwidth of 0.028 Hz [26]. Fig. 6(b) shows $\text{Im}\{\hat{S}_{UV}(f)\} \neq 0$, where $\hat{S}_{UU}(f)$ and $\hat{S}_{VV}(f)$ are nonzero, consistent with an elliptical

Fig. 5. Analysis of ULF wave in solar magnetic field February 2003. Component $\{U_t\}$ recorded parallel to ecliptic plane (solid) and component $\{V_t\}$ recorded perpendicular to ecliptic (dashed).

Fig. 6. Analysis of ULF wave. Components of $\hat{S}(f)$. (a) $\hat{S}_{UU}(f)$ (solid) and $\hat{S}_{VV}(f)$ (dotted). (b) $\text{Re}\{\hat{S}_{UV}(f)\}$ (solid) and $\text{Im}\{\hat{S}_{UV}(f)\}$ (dotted).

Fig. 7. Analysis of ULF wave in solar magnetic field. (a) Squared estimated DoP $\hat{P}^2(f)$, and corresponding critical value C^2 (dashed). (b) Unbiased squared DoP $\max\{0, \hat{Q}(f)\}$, with $\hat{Q}$ defined in (10).

polarization component [11]. There is no effective power in the signal beyond about 0.1 Hz. In view of this, the squared estimated DoP $\hat{P}^2(f)$ is shown in Fig. 7(a) for $f \leq 0.1$, along with the corresponding critical value C^2 when $\alpha = 0.01$. The frequency band for which $\hat{P}^2(f) > C^2$ is only a little larger than the effective analysis bandwidth of 0.028 Hz, suggesting a very narrow band signal centred at about 0.035 Hz. Finally, the unbiased squared DoP $\max\{0, \hat{Q}(f)\}$, with $\hat{Q}$ defined in (10), is shown in Fig. 7(b), and the effect of the unbiasing is mainly to “clean-up” the plot—smaller polarization values are reduced much more than larger values.

Fig. 8. Analysis of ULF wave in solar magnetic field February 2003. (a) Time evolution of U_t versus V_t . (b) Time evolution of U_t versus V_t after bandpass filtering to the interval $[0.033, 0.037]$ Hz.

The time evolution of the series U_t and V_t of Fig. 5 is shown in Fig. 8(a). If these series are bandpass filtered to the narrow band $[0.033, 0.037]$ Hz centered about the apparent carrier frequency of 0.035 Hz, the time evolution becomes the almost perfect ellipse shown in Fig. 8(b), confirming that the true ULF wave is elliptically polarized in this band.

VI. CONCLUDING COMMENTS

We have derived important and useful new statistical properties of the estimated DoP in the 2-D case. Formulas for the CDF, confidence intervals for P , and moments of $\hat{P}$ are provided and are readily computable. An exactly unbiased estimator of P^2 was also derived. Our statistical analyses identified elliptical polarization in a ULF wave in the solar magnetic field.

APPENDIX A

DISTRIBUTION FUNCTION OF $\hat{P}$

Let $c = 2(1 - P^2)^K / B(K - 1, 3/2)$. The cumulative distribution function of $\hat{P}$ is given by

$$\begin{aligned} F(p) &= c \int_0^p x^2 (1 - x^2)^{K-2} \\ &\quad \times {}_2F_1\left(K, K + \frac{1}{2}; \frac{3}{2}; P^2 x^2\right) dx \\ &= c \int_0^p x^2 (1 - x^2)^{K-2} \\ &\quad \times \sum_{n=0}^{\infty} \frac{(K)_n (K + \frac{1}{2})_n (P^2 x^2)^n}{(\frac{3}{2})_n n!} dx \\ &= c \sum_{n=0}^{\infty} \frac{(K)_n (K + \frac{1}{2})_n P^{2n}}{(\frac{3}{2})_n n!} \\ &\quad \times \int_0^p x^{2n+2} (1 - x^2)^{K-2} dx \\ &= \frac{c}{2} \sum_{n=0}^{\infty} \frac{(K)_n (K + \frac{1}{2})_n P^{2n}}{(\frac{3}{2})_n n!} \\ &\quad \times \int_0^{p^2} y^{n+\frac{1}{2}} (1 - y)^{K-2} dy. \end{aligned}$$

From [7, p. 666]

$$\int_0^{p^2} y^{b-1} (1 - y)^{a-1} dy = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \sum_{m=0}^{a-1} \frac{\Gamma(b+m)}{m!} (1 - p^2)^m$$

if $a \geq 1$ is an integer. Taking $a = K - 1$ and $b = n + 3/2$, we get

$$\begin{aligned} F(p) &= \frac{c}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{K-2} \frac{(K)_n (K + \frac{1}{2})_n \Gamma(K-1) \Gamma(n + \frac{3}{2})}{(\frac{3}{2})_n \Gamma(K + n + \frac{1}{2}) n! m!} \\ &\quad \times P^{2n} p^{2n+3} (1 - p^2)^m \end{aligned}$$

$$\begin{aligned}
&= \frac{c}{2} \sum_{m=0}^{K-2} \frac{\Gamma(K-1)\Gamma(m+\frac{3}{2})}{\Gamma(K+\frac{1}{2})m!} p^3 (1-p^2)^m \\
&\quad \times \sum_{n=0}^{\infty} \frac{\Gamma(K+n)\Gamma(m+n+\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(K)\Gamma(m+\frac{3}{2})\Gamma(n+\frac{3}{2})n!} (P^2 p^2)^n \\
&= \frac{c}{2} \sum_{m=0}^{K-2} \frac{\Gamma(K-1)\Gamma(m+\frac{3}{2})}{\Gamma(K+\frac{1}{2})m!} p^3 (1-p^2)^m \\
&\quad {}_2F_1\left(K, m+\frac{3}{2}; \frac{3}{2}; P^2 p^2\right).
\end{aligned}$$

Restoring the form of c we obtain, for $K \geq 2$ (with dependence on K and P shown explicitly)

$$\begin{aligned}
F(p; K, P) &= (1-P^2)^K p^3 \sum_{m=0}^{K-2} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(\frac{3}{2})m!} \\
&\quad \times (1-p^2)^m {}_2F_1\left(K, m+\frac{3}{2}; \frac{3}{2}; P^2 p^2\right).
\end{aligned}$$

APPENDIX B

EXACTLY UNBIASED ESTIMATOR FOR P^2

Using (7) the PDF $f(p)$, given in (6), can be rewritten as

$$\begin{aligned}
f(p) &= \frac{2}{B(K-1, \frac{3}{2})} \\
&\quad \times (1-P^2)^K p^2 (1-p^2)^{K-2} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(K)_n (K+\frac{1}{2})_n (Pp)^{2n}}{(\frac{3}{2})_n n!}.
\end{aligned}$$

Then, the unbiased estimator must satisfy

$$\begin{aligned}
1-P^2 &= E\{G(\hat{P})\} = \int_0^1 G(p)f(p)dp \\
&= \int_0^1 G(p) \frac{2}{B(K-1, \frac{3}{2})} \\
&\quad \times (1-P^2)^K p^2 (1-p^2)^{K-2} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(K)_n (K+\frac{1}{2})_n (Pp)^{2n}}{(\frac{3}{2})_n n!} dp \\
&= \sum_{n=0}^{\infty} \frac{2(K)_n (K+\frac{1}{2})_n}{B(K-1, \frac{3}{2}) (\frac{3}{2})_n n!} (1-P^2)^K P^{2n} \\
&\quad \times \int_0^1 G(p) p^{2n+2} (1-p^2)^{K-2} dp.
\end{aligned}$$

Rearranging, we see $(1-P^2)^{-K+1}$ can be written as

$$\sum_{n=0}^{\infty} P^{2n} \frac{2(K)_n (K+\frac{1}{2})_n}{B(K-1, \frac{3}{2}) (\frac{3}{2})_n n!} \int_0^1 G(p) p^{2n+2} (1-p^2)^{K-2} dp. \quad (15)$$

But by the negative binomial expansion

$$(1-P^2)^{-K+1} = \sum_{n=0}^{\infty} P^{2n} \frac{\Gamma(K+n-1)}{\Gamma(K-1)n!}. \quad (16)$$

Equating the coefficients of P^{2n} in (15) and (16), using $(y)_n = \Gamma(y+n)/\Gamma(y)$ and substituting $p = e^{-y/2}$, we get

$$\begin{aligned}
&\int_0^1 G(e^{-y/2})(1-e^{-y})^{K-2} e^{-3y/2} e^{-ny} dy \\
&= \frac{\Gamma(K+n-1)\Gamma(K)\Gamma(n+\frac{3}{2})}{\Gamma(K+n)\Gamma(K+n+\frac{1}{2})}.
\end{aligned}$$

The left side is the Laplace transform of $G(e^{-y/2})(1-e^{-y})^{K-2} e^{-3y/2}$ while the right side is the Laplace transform of $(1-e^{-y})^{K-1} e^{-3y/2} {}_2F_1(\frac{3}{2}, 1; K; 1-e^{-y})$ [6, p. 262, eq. (7)] so that

$$G(\hat{P}) = (1-\hat{P}^2) {}_2F_1\left(\frac{3}{2}, 1; K; 1-\hat{P}^2\right).$$

ACKNOWLEDGMENT

The authors would like to thank Dr. T. Horbury for supplying the Cluster mission time series, and the referees and associate editor for their helpful suggestions.

REFERENCES

- [1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1965.
- [2] M. Archer, T. S. Horbury, E. A. Lucek, C. Mazelle, A. Balogh, and I. Dandouras, "Size and shape of ULF waves in the terrestrial foreshock," *J. Geophys. Res.—Space Phys.*, vol. 110, p. A05208, 2005.
- [3] R. Barakat, "The statistical properties of partially polarized light," *Opt. Acta.*, vol. 32, pp. 295–312, 1985.
- [4] M. Born and E. Wolf, *Principles of Optics*, 4th ed. Cambridge, U.K.: Cambridge Univ. Press, 1970.
- [5] J. Calman, "On the interpretation of ocean current spectra. Part 1: The kinematics of three-dimensional vector time series," *J. Phys. Oceanog.*, vol. 8, pp. 627–643, 1978.
- [6] A. Erdelyi, *Tables of Integral Transforms, Volume 1*. New York: McGraw-Hill, 1954.
- [7] R. A. Fisher, "The general sampling distribution of the multiple correlation coefficient," *Proc. Roy. Soc. London A*, vol. 121, pp. 654–673, 1928.
- [8] N. R. Goodman, "Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction)," *Ann. Math. Stat.*, vol. 34, pp. 152–177, 1963.
- [9] J. W. Goodman, "Statistical properties of laser speckle patterns," in *Laser Speckle and Related Phenomena*, J. C. Dainty, Ed. New York: Springer-Verlag, 1975, pp. 9–75.
- [10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Corrected and Enlarged ed. New York: Academic, 1980.
- [11] Y. Hayashi, "Space-time spectral analysis of rotary vector series," *J. Atmosph. Sci.*, vol. 36, pp. 757–766, 1979.
- [12] J. Li, G. Yao, and L. V. Wang, "Degree of polarization in laser speckles from turbid media: Implications in tissue optics," *J. Biomed. Opt.*, vol. 7, pp. 307–312, 2002.
- [13] T. Medkour and A. T. Walden, "A variance equality test for two correlated complex Gaussian variables with application to spectral power comparison," *IEEE Trans. Signal Process.*, vol. 55, no. 3, pp. 881–888, Mar. 2007.
- [14] B. N. Nagarsenker and M. M. Das, "Exact distribution of sphericity criterion in the complex case and its percentage points," *Commun. Stat.*, vol. 4, pp. 363–74, 1975.
- [15] I. Olkin and J. W. Pratt, "Unbiased estimation of certain correlation coefficients," *Ann. Math. Stat.*, vol. 21, pp. 201–211, 1958.
- [16] J. Park, F. L. Vernon III, and C. R. Lindberg, "Frequency dependent polarization analysis of high-frequency seismograms," *J. Geophys. Res.*, vol. B92, pp. 12664–12674, 1987.
- [17] K. C. S. Pillai and B. N. Nagarsenker, "On the distribution of the sphericity test criterion in classical and complex normal populations having unknown covariance matrices," *Ann. Math. Stat.*, vol. 42, pp. 764–767, 1971.
- [18] J. C. Samson, "The reduction of sample bias in polarization estimators for multichannel geophysical data with anisotropic noise," *Geophys. J. R. Astron. Soc.*, vol. 75, pp. 289–308, 1983.
- [19] J. C. Samson and J. V. Olson, "Some comments on the description of the polarization states of waves," *Geophys. J. R. Astron. Soc.*, vol. 61, pp. 115–129, 1980.
- [20] V. S. del Rio, J. M. P. Mosquera, M. V. Isasa, and M. E. de Lorenzo, "Statistics of the degree of polarization," *IEEE Trans. Antennas Propag.*, vol. 54, pp. 2173–2175, 2006.
- [21] T. Setälä, A. Shevchenko, A. M. Kaivola, and A. T. Friberg, "Degree of polarization for optical near fields," *Phys. Rev. E*, vol. 66, no. 016615, Jul. 2002.

- [22] I. C. Sikaneta and J.-Y. Chouinard, "Eigendecomposition of the multi-channel covariance matrix with applications to SAR-GMTI," *Signal Process.*, vol. 84, pp. 1501–1535, 2004.
- [23] J. F. L. Simmons and B. G. Stewart, "Point and interval estimation of the true unbiased degree of linear polarization in the presence of low signal-to-noise ratios," *Astron. Astrophys.*, vol. 142, pp. 100–106, 1985.
- [24] R. J. A. Tough, D. Blacknell, and S. Quegan, "A statistical description of polarimetric and interferometric synthetic aperture radar data," *Proc. R. Soc. Lond. A*, vol. 449, pp. 567–589, 1995.
- [25] A. T. Walden, "A unified view of multitaper multivariate spectral estimation," *Biometrika*, vol. 87, pp. 767–787, 2000.
- [26] A. T. Walden, E. J. McCoy, and D. B. Percival, "The effective bandwidth of a multitaper spectral estimator," *Biometrika*, vol. 82, pp. 201–214, 1995.

Enumeration of Downsampling Lattices in Two-Dimensional Multirate Systems

Zhang Lei, *Student Member, IEEE*, and
Anamitra Makur, *Senior Member, IEEE*

Abstract—Lattices are essential for resampling in two-dimensional multirate systems. However, the resampling matrix for a given lattice is not unique. For a given resampling ratio, how many different lattices exist and how to parameterize them have not been addressed before. In this correspondence, such enumeration and parametrization are presented for the cases where the resampling ratio is prime or a composite number. The set of nonseparable resamplers is also studied. Examples and an application of the proposed enumeration and parametrization are included for demonstration.

Index Terms—Downsampling matrix, lattice, two-dimensional multirate system, unimodular matrix.

I. INTRODUCTION

Resampling, i.e., downsampling or upsampling, is essential to multirate systems. In a 1-D multirate system, resampling of a given ratio is usually performed in a unique pattern. This has been well studied [1]. However, for 2-D or higher dimension, the situation is more complicated. Even under the same (integer) resampling ratio, there may exist various ways or patterns to resample since samples are distributed along more than one axis. Many meaningful results have been presented in literature. Some are specifically focused on interchangeability of downsampler and upsampler [2]–[6]. Some have a more comprehensive scope [7]–[9]. According to these results, the nature of resamplers has been well understood.

Systems of the same resampling pattern may share many properties in common [1], [10]. For example, the filter bank that is alias free for a downsampling matrix is also alias free for other downsampling matrices of the same pattern after necessarily rearranging samples [1]. Exploration of seemingly infinite number of downsampling matrices can be reduced to exploration of a few distinct patterns. It is interesting to

Fig. 1. Three possible lattices for $|\det \mathbf{M}| = 2$.

ask how to enumerate and parameterize these distinct resampling patterns for a given resampling ratio. However, these questions have not been addressed in previous papers. This is precisely the scope of this work. Since upsampling is the dual of downsampling, in the following we limit our study to downsampling matrices without loss of generality. In this correspondence, we only focus on 2-D systems, which is of most practical importance and is a natural starting point for multidimensional system. In the literature, downsampling patterns are widely referred as lattices. Hence, we also stick to this notation.

Mathematically, given integer vectors $\mathbf{k}$ and $\mathbf{l}$, a lattice is a set of points $\mathcal{L}$ generated by a nonsingular integer matrix $\mathbf{M}$ according to

$$\mathcal{L} = \{\mathbf{l} | \mathbf{l} = \mathbf{M} \cdot \mathbf{k} \text{ for all } \mathbf{k} \in \mathbf{I}\}$$

where $\mathbf{I}$ is the set of all integer vectors. The downsampling ratio is correspondingly defined as the ratio of sample densities before and after downsampling. As an example, three possible lattices with downsampling ratio $m = 2$ are shown in Fig. 1. In 2-D systems, the downsampler is represented by a 2×2 nonsingular integer matrix, denoted as $\mathbf{M} = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$. According to [1], the corresponding downsampling lattice can be obtained by repeating vectors $[m_{00}, m_{10}]^T$ and $[m_{01}, m_{11}]^T$. The downsampling procedure retains only the samples that are on the generated lattice. Different lattices are represented by different downsampling matrices. In this example, the generating matrices might be $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. However, the reverse of above argument is not true. In fact, different matrices may generate the same lattice. A given lattice may be obtained from an infinite number of resampling matrices. These matrices are actually related by right multiplication of a unimodular matrix [1], which equivalently rearranges the samples before the original downsampling. In matrix language, downsampling matrices $\mathbf{M}$ and $\mathbf{M}'$ represent the same lattice if there exists a unimodular integer matrix $\mathbf{V}$ s.t. $\mathbf{M} = \mathbf{M}'\mathbf{V}$. Note that the unimodular matrix $\mathbf{V}$ is defined as a square matrix satisfying $|\det \mathbf{V}| = 1$. The aforementioned equivalence is denoted as $\mathbf{M} \leftrightarrow \mathbf{M}'$ in the following. However, left multiplication means rearrangement of samples afterwards, which will change the lattice.

We can enumerate different lattices by using downsampling matrices whose equivalence has been eliminated. Meanwhile, the obtained lattices without equivalence can be used for parametrization of lattices. Basically, the above principle is followed in this correspondence.

II. ENUMERATION AND PARAMETRIZATION OF LATTICES

A. Lattices With Prime Downsampling Ratio

A downsampling matrix is given by $\mathbf{M} = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$ with $m = |\det \mathbf{M}|$. The following equivalence is always obtainable.

Manuscript received September 8, 2006; revised May 7, 2007. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Zoran Cvetkovic.

The authors are with the School of Electrical & Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: zhanglei_eee3@pmail.ntu.edu.sg; EAMakur@ntu.edu.sg).

Digital Object Identifier 10.1109/TSP.2007.906688